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43
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MATHEMATICAL QUESTIONS AND SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES,"

AND THREE

APPENDICES.

EDITED BY

W. J. C. MILLER, B.A.,
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CONTENTS.

Mathematical Papers, &c.

Solved Questions.

MATHEMATICS

FROM

THE EDUCATIONAL TIMES.

WITH ADDITIONAL PAPERS AND SOLUTIONS.

2853. (Professor SYLVESTER, F.R.S.)—1. Let P be a given point in a cubic. Show that, in general, three conics may be drawn having contact of the 3rd order with the cubic of P, and ordinary contact with it at some other point; but that only two conics can be drawn when P is a pluperfect point of the 1st, 2nd, or 4th order.

2. Also that, in general, twelve conics may be drawn having ordinary contact with the cubic at P, and contact of the 3rd order with it at some other point; but that only eight such conics can be drawn when P is a pluperfect point of the 1st or 2nd order.

Solution by Professor NASH, M.A.

(For Notation see Questions 2352 and 2572, Vol. 45, pp. 126, 127.)

1. If Q is the point of ordinary contact, then Q is the point of contact of a tangent from P_2 (second tangential). There are three positions of Q exclusive of P_1, for which the conic would degenerate into the tangent PP_1 repeated, and therefore three conics.

(*a*) If P is pluperfect of the 1st order, *i.e.*, a point of inflexion, three points P of the conic are collinear, and the conic must break up into the tangent at P and a tangent from P. There are three such degenerate conics.

(*b*) If P is a pluperfect point of the 2nd order, or sextactic point, P_1 is a point of inflexion, and P_2 coincides with P_1. There are three tangents from P_2, one of which is P; the other two give two positions of Q and therefore two conics.

(*c*) The coresidual of 10 consecutive points on a cubic is found by taking two conics each through five consecutive points, to be the tangential of P_*, P_2. If the point P is pluperfect of the 4th order, a quartic can be described through 12 consecutive points, and the line joining the last two points must pass through $(P_*, P_2)_*$, (P_*, P_2), *i.e.*, this point must coincide with P_1; *i.e.*, if P_*, P_2 be R, then R_1, P_1 coincide.

Now P, P_2, R are collinear, therefore P_1, P_3, R_1 are collinear. But R_1 coincides with P_1, therefore P_3 coincides with P_2, therefore P_2 is a point of inflexion; therefore from P_2 there are only three tangents, one of them being P_2P_1, for which the conic degenerates; therefore, &c.

2. If the conic have simple contact at P, and four-point contact at Q, then Q_2 coincides with P_1, and is known. There are three positions of Q_1, exclusive of P, each of which gives four positions of Q, in all 12 conics. If Q_1 coincides with P, the conic degenerates into a tangent from P taken twice.

(*a*) If P is a point of inflexion, P_1 or Q_2 coincides with P, and there are three positions of Q_1, each of which gives four positions of Q, and therefore 12 conics.

(*b*) If P is a sextactic point, P_1 or Q_2 is a point of inflexion, and there are three positions of Q_1, one of which coincides with P; the other two give 8 positions of Q, and therefore 8 conics.

8861. (J. Brill, M.A.)—A rod of given length is broken at random into two pieces; find the probability that their lengths may be commensurable.

9588. (Charles L. Dodgson, M.A.)—A random point being taken on a given line, find the chance of its dividing the line into two parts (1) commensurable, (2) incommensurable.

Solutions by (1) Rev. T. C. Simmons, M.A.; (2) Prof. Tanner, M.A.

1. One method of dealing with the interesting Question 9588 is as follows:—It will be on all hands admitted that, if a random point P be taken on an undivided line λ, it is infinitely more likely to fall *between* the extremities than *on* either assigned extremity. Let λ now be divided, first into halves, then into thirds, quarters, fifths, sixths, and so on. At any stage of the operation suppose that n marks of division (including one of the original extremities of λ) have been in all obtained. The chance of P's falling *between* any two given consecutive marks will, as before, be infinitely greater than the chance of its falling *on* an assigned one of those two marks; so that its chance of falling *between some consecutive two* of the n marks, will be infinitely greater than its chance of falling *on some one* of the n marks. The ratio of the one chance to the other, having both its terms multiplied by n, will be independent of n, holding of course equally when n is made infinite. But, by making n infinite, all the possible commensurable divisions of λ can be apparently in time exhausted. Therefore the chance that P does not coincide with some possible commensurable division of λ is infinitely greater than the chance that it does so coincide; and on these grounds I venture the *opinion* that the answers to the question should be (1) zero, (2) unity.

In the same way the probability here required in Quest. 8861 would appear to be zero.

2. When Mr. Dodgson [see Note entitled *Something or Nothing*, on pp. 101, 2, of Vol. xlix.] explains how his two selected aggregates of points can make up the whole of a line consistently with his axiom (2), he will go far to help the "opposition" to explain how, notwithstanding axiom (1), an aggregate of absolute zeros may be unity.

Represent by δ the chance of a random point in a given line of unit length coinciding with an assigned point. Consider a range of points P, each distant δ/n from its neighbours. If δ is not absolute zero the points P are discrete. In a segment of length $1/n$ there will be $1/\delta$ of

these points, so that the chance of the random point coinciding with one or other of the points P is unity, and the chance of the random point falling outside the segment, however short, is zero. To avoid the absurdity, we must take $\delta = 0$.

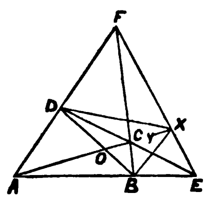
[In reference to the reasoning and arguments used in the above solutions, Mr. Dodgson sends the following remarks:—

(1) In reply to the Rev. T. C. Simmons, if, instead of dividing his line by 2, 3, &c., he will divide it by $\sqrt{2}$, $\sqrt{3}$, &c., and if, where he has written "commensurable," he will write "incommensurable," he will find his argument quite as sound as before, and, instead of proving the two chances to be "zero" and "unity," he will prove them to be "unity" and "zero." An argument that proves with equal ease either of two contradictories, needs very cautious handling.

(2) In reply to Professor Tanner, I must respectfully decline to explain how a thing can happen which I say cannot happen at all! No "aggregate of points," as I believe, can ever "make up the whole of a line," or any portion of it: so I must refer him, for the explanation he desires, to the "opposition," who are so ready to explain how "an aggregate of absolute zeros may be unity." While their hand is in, they may as well do the other little job. In the latter part of his letter he asserts, unless I misunderstand him, that, if the chance of a random point coinciding with one assigned point be δ, then its chance of coinciding with one or other of $1/\delta$ such points is unity. I suppose he would say, taking 10 bags, each containing 1 white counter and 9 black, that, since the chance of drawing a white from one bag is $\frac{1}{10}$, the chance of drawing a white from one or other of the 10 bags is unity. Does he accept this as a fair instance of the theorem?]

9544. (Professor Genese, M.A.) — O is the intersection of the diagonals of a quadrilateral inscribed in a circle, OX the perpendicular on the third diagonal. Prove that X is the intersection of the four circum-circles of the triangles determined by the sides of the quadrilateral.

Solution by W. S. Foster; R. Knowles, B.A.; *and others.*



Let ABCD be the quadrilateral; X the intersection of the circles EBC, EAD; XY perpendicular to EF; then EX, BC, AD meet in a point, for they are the radical axes of the three circles, ABCD, EBC, EAD; therefore X is on EF. And since X is a point on the circles circumscribing EBC, EAD, the feet of the perpendiculars from X on ED, EA, BC, AD lie on one straight line. Therefore X is also a point on the circles circumscribing FCD, FAB. Therefore X is the point of intersection of the four circles. Now EXDA FXBA are quadrilaterals inscribed in circles, the angle DXE = FXB. Therefore XY bisects the angle DXB, and it is perpendicular to EF,

therefore $\{X . BYDF\}$ is a harmonic pencil; but $\{E . BODF\}$ is a harmonic pencil, so is therefore $\{X . BODF\}$. Therefore XY passes through O, or OX is perpendicular to EF.

9562. (E. B. Elliott, M.A.)—**A cubic is described to pass through the three vertices of a triangle ABC, the three mid-points D, E, F of its sides, and its centroid G. Prove that (1) the tangents at A, B, C, G meet in a point P on the curve; that those at D, E, F, P meet in a second point Q on the curve; and that P, Q are the double points of the involution in which their connector is cut by the sides of the quadrangle whose vertices are A, B, C, G; and (2) if the cubic have one other point given, the locus of P is a straight line and that of Q a conic through D, E, and F.**

Solution by Professor Nash, M.A.

Every cubic through the seven given points cuts the given cubic in two other points X, Y, such that the line XY passes through a fixed point P on the cubic.

One such cubic is made up of the lines AGD, BGE, CGF; in this case X, Y both coincide with G; therefore the tangent at G passes through P.

Another cubic is made up of the lines AFB, AEC, AGD; X, Y both coincide with A, and the tangent at A passes through P. Hence A, B, C, G are the points of contact of the four tangents from a point P on the given cubic.

Since B, D, C are collinear, their tangentials are collinear; therefore P, Q, P are collinear, Q being tangential of D, and therefore Q is also tangential of P. Therefore tangents at D, E, F, P meet in a point Q on the curve.

Hence, by Salmon's *Higher Plane Curves*, Art. 153, P, Q are harmonic conjugates of the points in which the line PQ meets the pairs of chords joining the points of contact of tangents from P, *i.e.*, the sides of the quadrangle ABCG.

If the cubic is not fixed, but passes through an eighth fixed point H, then a ninth point K is also known, and the line HK must pass through P, *i.e.*, the locus of P is the line HK.

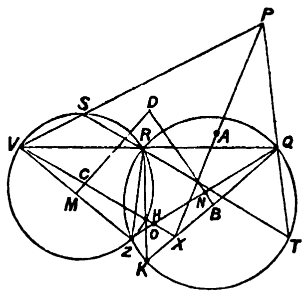
If D [P] denote the anharmonic ratio of the pencil formed by joining D to any four positions of P, then, since DP and DQ are harmonic conjugates of DA, DC, therefore D [P] = D [Q], and similarly, E [P] = E [Q]. But D [P] = E [P], since the P's are collinear; therefore D [Q] = E [Q]; therefore the four points Q lie on a conic through DE, and therefore also through F; therefore locus of Q is a conic through DEF.

The theorem, and method of proof, apply equally when G is any point, and DEF the intersection of AG, BG, CG with the opposite sides.

9431. (Professor Chakravarti, M.A.)—The straight lines TQP, TRS, VSP, and VRQ form four triangles; A, B, C, D are the centres of the circles circumscribing the triangles TPS, TQR, VRS, and VPQ re-

spectively: prove that the straight lines AP, BQ, and VC meet in a point which is the intersection of the circle VPQ with the circle ABCD.

Solution by Professor W. P. Casey; Fannie H. Jackson; *and others.*



The circumcircles of these four triangles intersect in a common point Z, and their centres A, B, C, D are concyclic. Let VC, QB meet in X. Join RH and produce it to meet the circle B in K. Then will QB produced pass through K, and

$$\angle ZVH = \angle ZRH = \angle ZQK;$$

therefore $\angle VZO = \angle OXQ$.

Whence circle VPQ will pass through X, and in like manner it will pass through the intersection of PA and QB, and therefore PA will meet QB in X. Join BD; it intersects QZ at right angles in N. And so does DC intersect VZ at right angles in M. Therefore MDNZ is concyclic, and therefore CDBX is so. Whence ABXCD is concyclic.

[PD, TB, SC will also meet in a point which is the intersection of the circle TPS with the circle ABCD, and the circle ABCD may be called the six-point circle.]

8655. (Asparagus.)—The circle of curvature at the point (XY) of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ touches an asymptote; prove that

$$\frac{X^4}{a^6}+\frac{Y^4}{b^6}=\pm\frac{2XY}{ab}\left(\frac{X^2}{a^4}+\frac{Y^2}{b^4}\right).$$

Solution by G. G. Storr, M.A.; Rev. T. Galliers, M.A.; *and others.*

The circle of curvature at (X, Y), and the condition that this circle shall touch one of the asymptotes $ay \pm bx = 0$, are (Salmon's *Conics*, 5th ed., Art. 251, Ex. 3) expressed by the equations

$$x^2+y^2-2\frac{a^2+b^2}{a^4}X^3.x-2\frac{a^2+b^2}{b^4}Y^3.y+X^2+Y^2+2\frac{a^4Y^2+b^4X^2}{a^2b^2}=0 \quad \text{...(1)},$$

$$(a^2+b^2)\left\{\frac{X^3}{a^3}\pm\frac{Y^3}{b^3}\right\}^2 = X^2+Y^2+2\frac{a^4Y^2+b^4X^2}{a^2b^2} \quad \text{............(2)}.$$

Putting $X = a\sec\phi$, $Y = b\tan\phi$, (2) becomes

$$a^2\sin^4\phi+b^2=\pm 2\sin\phi\,(b^2+a^2\sin^2\phi);$$

hence $\dfrac{a^4\sin^4\phi}{a^6}+\dfrac{b^4\tan^4\phi}{b^6}=\pm 2\,\dfrac{a\sec\phi\; b\tan\phi}{ab}\left(\dfrac{a^2\sec^2\phi}{a^4}+\dfrac{b^2\tan^2\phi}{b^4}\right)$,

which gives at once the result stated in the question.

9438. (F. R. J. Hervey.)—Prove that, (1) if four mutually orthocentric points be projected, two and two, in any order upon the asymptotes of any rectangular hyperbola passing through them, the four projections are mutually orthocentric; and (2), given four concyclic points, there is one pair only of rectangular axes such that, the points being projected as before, the projections are concyclic.

Solution by Professor Schoute.

1. Let $xy=k^2$ be the equation of one of the rectangular hyperbolas and $y=mx+p$ and $x=-my+q$, the equations of two lines perpendicular to one another, that pass through the four points. Then the projections of the two points situated on $y=mx+p$ on the axis x are determined by $mx^2+px-k^2=0$, the projections of the two points situated on $x=-my+q$ on the axis y by $my^2-qy+k^2=0$. Now the first part of the problem is a consequence of the fact that these two equations in x and y give to the product of the roots the same value, etc.

2. Suppose there are two rectangular axes that possess the desired relation to the four given points, and let the coordinates of these points with reference to these axes be (a, a'), (b, b'), (c, c') and (d, d'). Then the six relations hold $\quad ab=c'd',\quad ac=b'd',\quad ad=b'c',$

$$a'b'=cd,\quad a'c'=bd,\quad a'd'=bc.$$

Therefore $\quad aa'=bb'=cc'=dd'=k^2,\quad$ when $\quad k^4=abcd=a'b'c'd'.$

This proves that four points chosen at random will have the required position with respect to the asymptotes of the rectangular hyperbola through them, when the product of their distances to either asymptote is equal to k^4, k being the constant corresponding to that hyperbola. But the last condition is fulfilled, if the four points are concyclic. For the equation

$$\begin{vmatrix} a^2+\dfrac{k^4}{a^2}, & b^2+\dfrac{k^4}{b^2}, & c^2+\dfrac{k^4}{c^2}, & d^2+\dfrac{k^4}{d^2} \\ a, & b, & c, & d \\ \dfrac{k^2}{a}, & \dfrac{k^2}{b}, & \dfrac{k^2}{c}, & \dfrac{k^2}{d} \\ 1, & 1, & 1, & 1 \end{vmatrix}=0,$$

that is, the expression of the position of the points on a circle may be reduced to

$$\begin{vmatrix} a^4, & b^4, & c^4, & d^4 \\ a^3, & b^3, & c^3, & d^3 \\ a^2, & b^2, & c^2, & d^2 \\ a, & b, & c, & d \end{vmatrix}=k^4\begin{vmatrix} a^3, & b^3, & c^3, & d^3 \\ a^2, & b^2, & c^2, & d^2 \\ a, & b, & c, & d \\ 1, & 1, & 1, & 1 \end{vmatrix},$$

or $$abcd=k^4.$$

3919. (Professor HUDSON, M.A.)—A man's expenses exceed his income by £a per annum; he borrows at the end of every year enough to meet this, and after the first year, to pay the interest on his previous borrowings, the rate of interest at which he borrows increasing each year in geometrical progression whose common ratio is λ, till at the end of n years it is cent. per cent. What does he then borrow?

Solution by D. BIDDLE.

Let μ = the original rate of interest, divided by 100. Then the sums borrowed year by year increase as follows:—

$$a, \quad a+\mu a, \quad a+(a+\mu a)\lambda\mu,$$

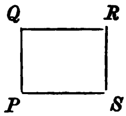
$$a+\left\{a+(a+\mu a)\lambda\mu\right\}\lambda^2\mu, \quad a+\left[a+\left\{a+(a+\mu a)\lambda\mu\right\}\lambda^2\mu\right]\lambda^3\mu, \ \&c.$$

At the end of the nth year, the factor representing the rate of interest will be $\lambda^{n-1}\mu$; and, since $\lambda^{n-1}\mu = 100/100 = 1$, therefore $\mu = 1/\lambda^{n-1}$, and the final sum borrowed is

$$a\left(1+1+\frac{1}{\lambda}+\frac{1}{\lambda^{1+2}}+\frac{1}{\lambda^{1+2+3}}+\frac{1}{\lambda^{1+2+3+4}}+\ldots+\frac{1}{\lambda^{1+2+\ldots+(n-1)}}\right).$$

9329. (R. F. DAVIS, M.A.)—PQRS is a path bounding a square garden. A and B walk backwards and forwards at uniform rates between P and R and S and R respectively. They start simultaneously from P and S, walking in parallel directions. Find, when A is at R for the n^{th} time, the distance of B from him.

Solution by the PROPOSER.



Suppose t, t' are the number of minutes it takes A and B respectively to walk one side of the square; then in $(4n-2)\,t$ minutes A is at R for the n^{th} time. At the same moment B will have walked $(4n-2)\,t/t'$ times the side of the square. Now, if x be the fractional part of the side of the square representing the distance of B from R at this moment, $(4n-2)\,t/t'$ = odd number $+x$, or even number $+(1-x)$ = odd number $+x$; hence

$$x = \left\{(4n-2)\,t/t' \sim (\text{nearest odd number thereto})\right\}.$$

9520. (A. KAHN.)—Solve the equations

$$xy+yz+zx = 47, \quad x(y+z-x)+y(z+x-y)+z(x+y-z) = 44,$$

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} = \frac{65}{42}.$$

Solution by A. E. Thomas; E. F. Elton, M.A.; *and others.*

Combining (1) with (2), we find $\Sigma x = \pm 12$. From (3), we have

$$\frac{(x+y+z)\left[(\Sigma x)^2+\Sigma xy\right]}{\Sigma x \,.\, \Sigma xy - xyz} = 3 + \tfrac{4}{1}\tfrac{7}{5}, \text{ giving } xyz = \pm 60.$$

Thus, x, y, z are the roots of $t^3 \mp 12t^2 + 47t \mp 60 = 0$, giving two sets of values $x = 3$, $y = 4$, $z = 5$, etc., etc.; $x = -3$, $y = -4$, $z = -5$, etc.

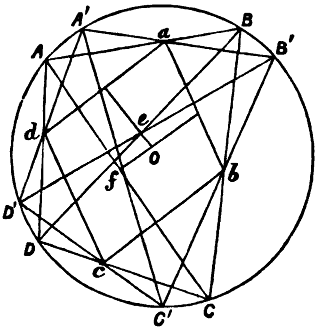
9488. (C. Bickerdike.)—If O be a fixed and Q a variable point on a circle whose centre is C, and if OQ is produced to P so that QP = 2OC; prove that the radius of curvature of the locus P is $\frac{4}{3}(\text{OC}\,.\,\text{OP})^{\frac{1}{2}}$.

Solution by Professor Scott, M.A.; R. Knowles, B.A.; *and others.*

The circle, taking O as origin and OC as prime radius vector, is $r = 2a\cos\theta$; hence the locus of P is $r = 2a(1+\cos\theta) = 4a\cos^2\frac{1}{2}\theta$. This becomes $r^3 = 4ap^2$, where p is the perpendicular on the tangent let fall from O; hence $\rho = \dfrac{rdr}{dp} = \dfrac{8ap}{3r} = \dfrac{32a^2\cos^3\frac{1}{2}\theta}{3r} = \frac{4}{3}a^{\frac{1}{2}}r^{\frac{1}{2}}$ as required, since a = OC and r = OP.

9508. (Professor Ignacio Beyens.)—Si un quadrilatère ABCD inscrit dans une circonference tourne autour du centre O de cette circonference jusqu'à la position A′B′C′D′, et (a) est la rencontre de AB, A′B′; (b) de BC, B′C′; (c) de CD, C′D′; (d) de DA, D′A′; (e) de BD, B′D′; (f) de AC, A′C′: démontrer que $abcd$ est un parallélogramme, et que les côtés ab, ad sont respectivement perpendiculaires à Of, Oe.

Solution by Professor De Wachter.



Let OA, OB, OC, OD be denoted by α, β, γ, δ. If i represents the unit-vector which turns another through a right angle in a certain direction about the centre O, then i^n will be the amount of rotation through an angle n in the same direction, and thus we have

$\text{OA}' = i^n\alpha$, $\text{OB}' = i^n\beta$, $\text{OC}' = i^n\gamma$,

$\text{OD}' = i^n\delta$. $\dfrac{\text{A}a}{\text{AB}}$ being a scalar $= x$,

we get, from the quadrilaterals OAA′a and OBB′a,

$$\text{O}a = \alpha + x(\beta - \alpha)$$
$$= \alpha i^n + (1-x)(\beta-\alpha)\,i^n.$$

Hence, by elimination of x,

$$Oa = (\alpha+\beta)\frac{i^n}{1+i^n}. \quad \text{Similarly,} \quad Ob = (\beta+\gamma)\frac{i^n}{1+i^n},$$

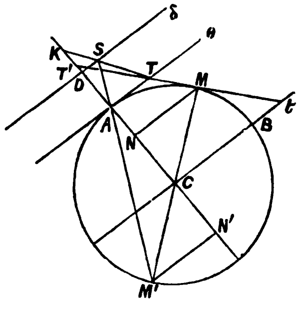
$$Oc=(\gamma+\delta)\frac{i^n}{1+i^n},\quad Od=(\delta+\alpha)\frac{i^n}{1+i^n},\quad Of=(\alpha+\gamma)\frac{i^n}{1+i^n},\quad Oe=(\beta+\delta)\frac{i^n}{1+i^n}.$$

From these vector-equations it follows at once:—(1) That ab and dc are equal vectors, since $Oa-Ob = Od-Oc$, therefore $abcd$ is a parallelogram; (2) that $ab=Ob-Oa=(\alpha-\gamma)\frac{i^n}{1+i^n}$ is perpendicular to $Of=(\alpha+\gamma)\frac{i^n}{1+i^n}$, since α and γ are vectors of the same length. Finally, ad is perpendicular to Oe, the reason being the same.

[*Otherwise*:—By the theorem in Question 9630, the point O is the orthocentre of the triangles ade, bce; hence Oe is perpendicular to ad, and, similarly, Of perpendicular to ab.]

9553. (MAURICE D'OCAGNE.)—A étant un point fixe pris sur une conique, θ la tangente en ce point, δ une parallèle quelconque à θ, si M et M′ sont les extrémités d'un diamètre variable de la conique, que la tangente en M à la conique coupe la tangente θ au point T et que la droite AM′ coupe la droite δ au point S, la droite ST passe par un point fixe du diamètre de la conique qui aboutit au point A. Quand la droite δ est rejetée à l'infini le point fixe est le centre de la conique.

Solution by W. S. FOSTER; BELLE EASTON, B.Sc.; *and others.*



Let the tangent at M meet the diameter CA in T′, SD the line δ; then

$$DS : AD = M'N' : AN',$$
$$DK : DS = AK : AT;$$

hence we have

$$DK : AD = MN \,.\, AK : AN' \,.\, AT$$
$$= AK \,.\, NT' : AN' \,.\, AT'$$
$$= AK \,.\, NT' : AC \,.\, NT'$$
$$= AK : AC;$$

thus it follows that

$$AK : AD = AC : AC-AD;$$

therefore K is a fixed point, and, when SD is moved to infinity, K coincides with C.

9483. (A. RUSSELL, B.A.)—Prove that the volume of the solid contained between the planes $z = a+k$, $z = a$, and the surfaces

$$x^2+y^2-z^2 = 0, \quad y^2(x^2+y^2+z^2) = x^2(z^2-x^2-y^2),$$

is
$$(\pi-1)\,ak\,(a+k)+\tfrac{1}{3}(\pi-1)\,k^3.$$

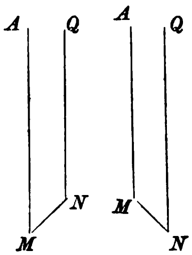
Solution by the Proposer; Prof. Ignacio Beyens; *and others.*

The second surface may be written in the form $(x^2+y^2)^2 = z^2(x^2-y^2)$, and the section by any plane parallel to the plane of xy is a lemniscate, the surface being got by joining all points on this curve to the origin. Thus volume cut off by the plane $z = a+k$ is $\frac{1}{3}(a+k)^3$, and volume cut off cone by this plane is $\frac{1}{3}\pi(a+k)^3$. Thus required volume

$$= \tfrac{1}{3}(\pi-1)\left\{(a+k)^3-a^3\right\} = (\pi-1)\,ak\,(a+k)+\tfrac{1}{3}(\pi-1)\,k^3.$$

8477. (Professor Swaminatha Aiyar, M.A.) — A system of rods AB, BC, CD ... PQ, freely jointed at B, C, D, ... Q, and of lengths 1, 3, 5, 7, 9 ..., is suspended from a smooth horizontal wire which passes through two rings at A and Q. Find how many rods the system must consist of, in order that in the position of equilibrium one of them may be inclined to the vertical at $\tan^{-1}\frac{7}{24}$.

Solution by the Proposer.



It is easily seen that in the position of equilibrium the system must consist of three lines, two of which are vertical, the third consisting of a single rod connecting the other two as in the annexed figures.

Let x be the total number of rods, and y the number of rods in AM. Then we should have

$$\frac{y^2+(y+1)^2-x^2}{2y+1} = \frac{24}{25} \quad\text{..................(1)},$$

or

$$\frac{x^2-y^2-(y+1)^2}{2y+1} = \frac{24}{25} \quad\text{..................(2)}.$$

$x = 17$, $y = 12$, satisfies the first equation.

8483. (R. Rawson, F.R.A.S.)—Solve, by the ordinary Riccatian process, the equation $\dfrac{d^2u}{dx^2} \pm \dfrac{2i+(i\mp\frac{1}{2})\,m}{x}\dfrac{du}{dx} + bcx^m w = 0.$

Solution by D. Edwardes; *the* Proposer; *and others.*

Transforming the equation $\dfrac{d^2u}{dx^2}+\dfrac{A}{x}\dfrac{du}{dx}+bx^m u = 0$ (A a constant)

by $$u = \exp.\int\frac{y+C}{x}\,dx,$$

we have $$x\frac{dy}{dx}-(1-2C-A)\,y+y^2+C^2-(1-A)\,C+bx^{m+2} = 0.$$

Put $C^2-(1-A)C = 0$, therefore $C = 0$ or $1-A$. Let $C = 0$, then

$$x\frac{dy}{dx}-(1-A)y+y^2+bx^{m+2} = 0,$$

which is integrable in finite terms if

$$\frac{m+2\pm 2(1-A)}{2m+4} = i \text{ a positive integer};$$

whence $A = \pm\left\{2i+(i\mp\frac{1}{2})m\right\}$; and $C = 1-A$ leads to the same condition. The equation in y can then be solved as in BOOLE's *Differential Equation*, Chap. VI.

[In a paper ("On RICCATI's Equation and its Transformations, and on some definite Integrals which satisfy them") given in the *Phil. Trans. of the Royal Society*, Part III., 1881, Mr. J. W. L. GLAISHER states that M. LEBESQUE has solved by general differentiation the equation

$$\frac{d^2y}{dx^2}\pm\frac{2i}{x}\frac{dy}{dx}+hy = 0,$$

where i is a positive integer and h a constant. The equation in the Question is a more general form of this.]

9216. (A. R. JOHNSON, M.A.)—Investigate the induced magnetization of an ellipsoidal shell of varying material, the surfaces of constant magnetic inductive capacity being confocals.

Solution by the PROPOSER.

The solenoidal condition gives for the total potential V, using the usual notation,

$$\frac{1}{h_1}\frac{d}{d\epsilon}\left(\frac{h_2h_3}{h_1}\mu\frac{dV}{d\epsilon}\right)+\frac{1}{h_2}\frac{d}{dv}\left(\frac{h_3h_1}{h_2}\mu\frac{dV}{dv}\right)+\frac{1}{h_3}\frac{d}{dv'}\left(\frac{h_1h_2}{h_3}\mu\frac{dV}{dv'}\right) = 0 \ldots (a).$$

The proper assumption as to the form of V in the material of the shell is $V = f(\epsilon)S$, ES being the inducing potential.

Substituting in (a), $\quad 0 = \dfrac{1}{h_1}\dfrac{d}{d\epsilon}\left(\dfrac{h_2h_3}{h_1}\mu f'(\epsilon)\right)S+\mu f(\epsilon)H$ (say),

and we know that $\quad 0 = \dfrac{1}{h_1}\dfrac{d}{d\epsilon}\left(\dfrac{h_2h_3}{h_1}\dfrac{dE}{d\epsilon}\right)S+EH,$

therefore $\quad \dfrac{d}{d\epsilon}\left(\dfrac{h_2h_3}{h_1}\mu f'(\epsilon)\right)\Big/\mu f(\epsilon) = \dfrac{d}{d\epsilon}\left(\dfrac{h_2h_3}{h_1}\dfrac{dE}{d\epsilon}\right)\Big/E \quad \ldots\ldots\ldots\ldots (b).$

But when $V = a$ in (a), and $\mu = 1$, (a) is satisfied; therefore

$$\frac{h_2h_3}{h_1}\frac{da}{d\epsilon} = \text{function of } v \text{ and } v' \text{ only},$$

and (b) becomes $\quad \dfrac{d}{d\epsilon}\left(\dfrac{\mu\dot{f}}{\dot{a}}\right)\Big/\mu f = \dfrac{d}{d\epsilon}\left(\dfrac{\dot{E}}{\dot{a}}\right)\Big/E,$

or $\quad \dfrac{d}{da}\left(\mu\dfrac{df}{da}\right)\Big/\mu f = \dfrac{d^2E}{da^2}\Big/E \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (c).$

But the right-hand member is $i\,(i+1)\,\epsilon^2 - z$; hence we have, for the determination of f, $\frac{d}{da}\left(\mu\frac{df}{da}\right) + \{z - i\,(i+1)\,\epsilon^2\}\,\mu f = 0$ ....(*d*).

Choose for the internal and external potentials

$$V_i = A\frac{E}{E_1},\quad V_e = B\frac{F}{F_2} + E.$$

Then, for the determination of the constants of f and of A, B, we have the surface-conditions

$$f_1 = A,\ f_2 = B + E_2;\quad \mu_1\dot{f}_1 = \mu_i A\frac{\dot{E}_1}{E_1},\quad \mu_2\dot{f}_2 = \mu_e\left(B\frac{\dot{F}_2}{F_2} + \dot{E}_2\right).$$

By attaching suitable values to μ we may derive some easily soluble cases; *e.g.*, let $\qquad \mu = [k^2/\{z - i\,(i+1)\,\epsilon^2\}]^{\frac{1}{2}}.$

Then, after some reductions, we get, in the material of the shell

$$V = \left\{\mu_i\frac{\dot{E}_1}{E_1}\sin\left[k\int_{\epsilon_1}^{\epsilon}\frac{da}{\mu}\right] + k\cos\left[k\int_{\epsilon_1}^{\epsilon}\frac{da}{\mu}\right]\right\}X,\ \text{and}\ V_i = kX,$$

$$V_e = E_2\left[\frac{\dot{E}}{E_2} - \frac{F}{F_2}\right]S + \frac{F}{F_2}\left\{\mu_i\frac{\dot{E}_1}{E_1}\sin\left[k\int_{\epsilon_1}^{\epsilon_2}\frac{da}{\mu}\right] + k\cos\left[k\int_{\epsilon_2}^{\epsilon_1}\frac{da}{\mu}\right]\right\}X,$$

the other factor in each case being

$$S\,\frac{\mu_e}{E_2}\left(\frac{E_2}{F_2} - \frac{\dot{E}_2}{\dot{F}_2}\right)$$

$$\Big/\left\{\left(\mu_i\mu_e\frac{\dot{E}_1\dot{F}_2}{E_1F_2} - k^2\right)\sin\left[k\int_{\epsilon_1}^{\epsilon_2}\frac{da}{\mu}\right] + k\left(\mu_i\frac{\dot{E}_1}{E_1} + \mu_e\frac{\dot{F}_2}{F_2}\right)\cos\left[k\int_{\epsilon_1}^{\epsilon_2}\frac{da}{\mu}\right]\right\}.$$

9491. (H. L. Orchard, M.A., B.Sc.)—A right-angled triangle has its angles in arithmetical progression. Show that the line that joins the right angle to the middle of the hypoteneuse divides the triangle into two equal triangles, the one equilateral, and the other isosceles.

Solution by R. W. D. Christie; R. Knowles, B.A.; *and others.*

The triangles are equal since they are upon equal bases and between the same parallels.

Since the angles are in A.P., they are 30°, 60°, 90°.

The rest easily follows.

9531. (H. L. Orchard, M.A., B.Sc.)—Show, by a simple quadratic method, that the roots of the equation

$$x^6 + (bx^{\frac{1}{2}} - x^{\frac{3}{2}})^4 - ax^2 = (b-x)^4 + x^4 - a$$

are ± 1, and $\frac{1}{2}\left\{b \pm \left[-3b^2 \pm 2\,[2\,(a+b^4)]^{\frac{1}{2}}\right]^{\frac{1}{2}}\right\}$.

Solution by E. F. Elton, M.A.; *the* Proposer; *and others.*

The equation may be put in the form

$$(x^2-1)\left[(4x^2-4bx+b^2)^2+6b^2(4x^2-4bx+b^2)+b^4-8a\right]=0;$$

hence $$x=\pm 1,$$

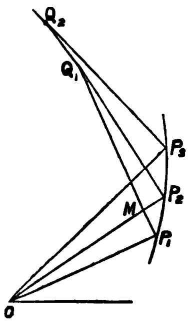
and $$(4x^2-4bx+b^2)^2+6b^2(4x^2-4bx+b^2)+b^4-8a=0,$$

$$4x^2-4bx+b^2=-3b^2\pm 2\sqrt{2}\,(b^4+a),$$

whence the stated result follows.

9455. (Professor Abinash Chandra Basû, M.A.)—At each point of a curve whose equation is $r=f(\theta)$ lines are drawn at right angles to the radius-vector to the point; prove that, if s be the arc measured from a fixed point of the envelope of these lines (the first negative pedal), then, between proper limits, $\mathrm{S}=\int f''(\theta)\,d\theta+\int f(\theta)\,d\theta.$

Solution by J. McMahon; Professor Kalipada Basû; *and others.*



Let the vectors to the consecutive points P_1, P_2, P_3 be represented by r, $(1+\Delta)\,r$, $(1+\Delta)^2\,r$; let Q_1, Q_2, Q_3 be the consecutive intersections of the perpendiculars at their extremities; then P_1Q_1, P_2Q_2. P_3Q_3 may be represented by t, $(1+\Delta)\,t$, $(1+\Delta)^2\,t$. Let P_1Q_1 meet OP_2 in M. Let any infinitesimal of the k^{th} order be denoted by i_k whenever its absolute magnitude is not needed.

Then $\quad \mathrm{OM}=r\sec\Delta\theta=r\,(1+\tfrac{1}{2}\Delta\theta^2+i_4),$

$\therefore\quad \mathrm{MP}_2,=(1+\Delta)\,r-\mathrm{OM},=\Delta r-\tfrac{1}{2}r\Delta\theta^2+i_4;$

but $\quad \mathrm{MP}_2=\mathrm{MQ}_1\tan\Delta\theta=\mathrm{MQ}_1\Delta\theta+i_3,$

therefore $\mathrm{MQ}_1=\dfrac{\mathrm{MP}_2}{\Delta\theta}+i_2=\dfrac{\Delta r}{\Delta\theta}-\tfrac{1}{2}r\,\Delta\theta+i_2;$

but $\quad \mathrm{P}_1\mathrm{M}=r\sin\Delta\theta=r\Delta\theta+i_3,$

therefore $\quad t,=\mathrm{MQ}_1+\mathrm{P}_1\mathrm{M},=\dfrac{\Delta r}{\Delta\theta}+\tfrac{1}{2}r\,\Delta\theta+i_2,$

therefore $\quad \mathrm{P}_2\mathrm{Q}_2,=(1+\Delta)\,t,=\dfrac{\Delta r}{\Delta\theta}+\tfrac{1}{2}r\,\Delta\theta+\Delta\,\dfrac{\Delta r}{\Delta\theta}+i_2;$

but $\quad \mathrm{P}_2\mathrm{Q}_1,=\mathrm{MQ}_1+i_2,=\dfrac{\Delta r}{\Delta\theta}-\tfrac{1}{2}r\,\Delta\theta+i_2,$

therefore $\quad \mathrm{Q}_1\mathrm{Q}_2=r\,\Delta\theta+\Delta\,\dfrac{\Delta r}{\Delta\theta}+i_2;$

but $\quad \Delta\mathrm{S}\sim\mathrm{Q}_1\mathrm{Q}_2<l_3,\qquad$ [Williamson, page 38]

therefore $\quad \dfrac{\Delta\mathrm{S}}{\Delta\theta}=r+\dfrac{\Delta}{\Delta\theta}\cdot\dfrac{\Delta r}{\Delta\theta}+i_1,$

and proceeding to the limit, we have

$$\frac{ds}{d\theta}=r+\frac{d^2r}{d\theta^2}=f\theta+f''\theta,\quad\text{therefore } \mathrm{S}=\int f\theta\,d\theta+\int f''\theta\,d\theta,$$

between proper limits.

8897. (R. Knowles, B.A.)—In Quest. 8753 prove that the circles ABD, ACD intersect at right angles in the point A.

Solution by Professor Ignacio Beyens ; Sarah Marks, B.Sc. ; *and others.*

Soient O, O′ les centres des cercles ABD, ACD. Nous avons

$$\text{OAD} = 90^\circ - \text{B}, \quad \text{O}'\text{AD} = 90^\circ - \text{C};$$

donc $$\text{OAO}' = 180^\circ - (\text{B}+\text{C}) = 180^\circ - 90^\circ = 90^\circ,$$

parce que le triangle ABC est rectangle en A, et OAO′ étant égal à 90° la proposition est démontrée.

8878. (Professor Wolstenholme, M.A., Sc.D.)—Prove that

$$\int_0^\infty \frac{\log(1+n)}{x^{1-n}}\,dx = \frac{\pi}{n\sin n\pi}\ (n > 0 < 1) \ldots\ldots\ldots\ldots (1),$$

$$\int_0^\infty \frac{x^{n-m-1}+x^{n-m-1}}{x^{2n}+2x^n\cos na+1}\,\frac{dx}{1\pm x^p} = \frac{\pi}{n\sin(m\pi/n)}\,\frac{\sin ma}{\sin na} \ldots\ldots\ldots (2),$$

p being any real number, m positive and $< n$, $n > 1$, and na lying between $-\pi$ and π; and (3) p being any real number,

$$\int_0^\infty F\left(x+\frac{1}{x}\right)\frac{1}{1\pm x^p}\,\frac{dx}{x} = \tfrac{1}{2}\int_0^\infty F\left(x+\frac{1}{x}\right)\frac{dx}{x} = \int_0^1 F\left(x+\frac{1}{x}\right)\frac{dx}{x}.$$

Solutions by the Proposer ; D. Edwardes ; *and others.*

1. $$\int_0^\infty \frac{\log(1+x)}{x^{1+n}}\,dx = -\frac{x^{-n}}{n}\log(1+x)\Big|_0^\infty + \frac{1}{n}\int_0^\infty \frac{dx}{x^n(1+x)};$$

the integrated part vanishes at both limits if $n > 0 < 1$,

and $$\int_0^\infty \frac{x^{-n}\,dx}{1+x} = \frac{1}{1-n}\int_0^\infty \frac{dz}{1+z^{1/(1-n)}},\ (z = x^{1-n})$$

$$= \frac{1}{1-n}\,\frac{\pi}{1/(1-n)\sin(1-n)\pi} = \frac{\pi}{\sin n\pi}.$$

2. If u be the proposed integral, and if we transform it by writing $1/x$ for x throughout, we get

$$u = \int_0^\infty \frac{x^{n+m-1}+x^{n-m-1}}{1+2x^n\cos na+x^{2n}}\cdot\frac{\pm x^p}{1\pm x^p}\cdot dx,$$

whence $$2u = \int_0^\infty \frac{x^{n-m-1}+x^{n-m-1}}{1+2x^n\cos na+x^{2n}}\,dx = 2\int_0^\infty \frac{x^{n-m-1}\,dx}{1+2x^n\cos na+x^{2n}},$$

or $$u = \frac{\pi}{n\sin m\pi/n}\,\frac{\sin ma}{\sin na},$$ under the conditions stated.

[*Math. Problems*, 1919, (50)].

3. If u be the proposed integral, and if we transform it by writing $1/x$ for x throughout, we get

$$u = \int_0^\infty \mathrm{F}\left(\frac{1}{x} + x\right) \frac{\pm x^p}{1 \pm x^p} \frac{dx}{x},$$

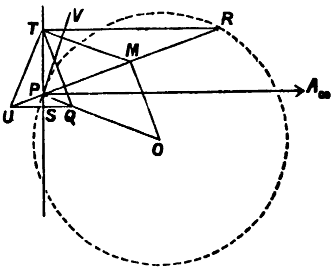
so that

$$2u = \int_0^\infty \mathrm{F}\left(x + \frac{1}{x}\right) \frac{dx}{x} = \int_0^1 \mathrm{F}\left(n + \frac{1}{x}\right) \frac{dx}{x} + \int_1^\infty \mathrm{F}\left(x + \frac{1}{x}\right) \frac{dx}{x},$$

and using the same transformation as before on the second term, it coincides with the first, or $u = \int_0^1 \mathrm{F}\left(x + \frac{1}{x}\right) \frac{dx}{x}$.

9445. (R. Knowles, B.A.)—If Q be the point through which pass all chords of a parabola that subtend a right angle at P, M the mid-point of the chord of curvature at P, and O the centre of curvature at P; prove that PM = OQ.

Solution by Professor Schoute; W. S. Foster; *and others.*



When the parabola is given by P, the centre of curvature O and the chord of curvature PR at P, the internal bisector of angle OPR is a diameter. The second point S of the parabola on the line PS perpendicular on this bisector PA_∞ is found by means of the line RT perpendicular to PS, TU parallel to the tangent in P, and US perpendicular to TS (TU is the Pascal line for the inscribed hexagon $PPRA_\infty A_x S$). Now Q is found by producing US, but then US = SQ and TQ is antiparallel to TU with reference to TS. Hence TQ is perpendicular to PR and parallel to MO. And TM and QO are antiparallel to PR with respect to TR. This proves TMOQ to be a parallelogram; therefore PM = TM = QO.

9498. (Professor Byomakesa Chakravarti, M.A.)—Prove that a triangle ABC is equilateral if cot A + cot B + cot C = $\sqrt{3}$.

Solution by A. M. Williams, M.A., *and* Rev. J. L. Kitchin, M.A.

This becomes, using the ordinary values of cosines and sines,

$$(a^2+b^2+c^2)^2 = 3(a+b+c)(b+c-a)(c+a-b)(a+b-c),$$

which is equivalent to $(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 = 0$.

Hence $a = b = c$. See Question 9496.

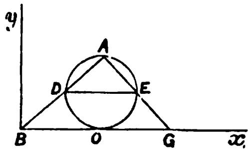
[Since $\cot A + \cot B + \cot C = \cot \omega$, and $\rho' = \frac{1}{2}R\,(1 - 3\tan\omega^2)^{\frac{1}{2}}$, we have $\rho' = 0$; thus the BROCARD-circle becomes a point; hence the circumcentre and the LEMOINE-point coincide; consequently we have

$$\cos A : \cos B : \cos C = \sin A : \sin B : \sin C, \quad \cot A = \cot B = \cot C,$$

and $$A = B = C = 60^\circ.]$$

8893. (J. BRILL, M.A.)—A circle is drawn passing through the vertex A of the triangle ABC, touching the base BC at O, and meeting the sides AB, AC in D, E; prove that $BC : DE = AB \cdot AC : OA^2$.

Solution by G. G. STORR, M.A.; R. KNOWLES, B.A.; *and others.*



Let $BO = h$, then the equation to the circle is

$$x^2 + y^2 - 2hx - \frac{c^2 - 2ch\cos B + h^2}{c\sin B}\,y + h^2 = 0;$$

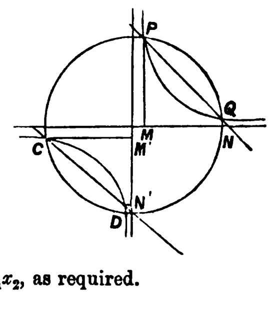
hence its radius

$$= \frac{c^2 - 2ch\cos B + h^2}{2c\sin B} = \frac{AO^2}{2c\sin B} = \frac{DE}{2\sin A},$$

which is the result in the Question.

9470. (R. KNOWLES, B.A.)—PQ, CD are common chords of a circle and rectangular hyperbola; PM, QN are perpendiculars to one asymptote, CM′, DN′ to the other; prove that $PM \cdot QN = CM' \cdot DN'$.

Solution by Professors A. W. SCOTT; SARKAR; *and others.*



Let $\quad xy = a^2$ be the hyperbola;

$$lx + my = 1, \qquad \lambda x + \mu y = 1,$$

equations of PQ and CD.

$$xy - a^2 + k\,(lx + my - 1)(\lambda x + \mu y - 1) = 0$$

must be the circle, hence $l\lambda = m\mu$ ...... (1),

From $xy - a^2 = 0$ and $lx + my - 1 = 0$, we get

$$my^2 - y + a^2 l = 0;$$

hence $\quad y_1 y_2 = a^2 l/m = PM \cdot QN,$

also $\quad \lambda x^2 - x + a^2\mu = 0,$

$$x_1 x_2 = a^2\mu/\lambda = CM' \cdot DN'.$$

But from (1) $\; l/m = \mu/\lambda$; hence $y_1 y_2 = x_1 x_2$, as required.

Notes on the Recent Geometry of the Triangle.

By the Editor.

The publication, in the *Educational Times* for May, 1888, of my Review of Mr. Milne's *Companion*—which contains an excellent *Introduction to the Recent Geometry of the Triangle*, by the Rev. T. C. Simmons—has procured for me many letters in regard to the origin of what is sometimes termed the *Brocardian Geometry*. As some of these letters contain historical matter of permanent interest, I purpose, now and hereafter, to give such abridged extracts from them as our very restricted space will allow.

1. Professor Emmerich furnishes, from his own *Notes*, the following data :—

"Mr. Simmons's *Introduction* affords an eloquent testimony to the interest which the pursuit of the historical development of the Recent Geometry finds in your country. I conclude, therefore, that some new historical observations on this subject will be not unwelcome to your readers.

Dr. A. L. Crelle, founder of *Crelle's Journal*, treats in his pamphlet 'On some properties of the Plane Triangle with regard to three straight lines drawn through the angular points' (Berlin, 1816), in section 12, the proposition—'Within a triangle ABC to find the point M for which $\angle MAB = MBC = MCA$'; and, by means of Ceva's theorem, arrives at the results

$$\sin\kappa^3 = \sin(A-\kappa)\sin(B-\kappa)\sin(C-\kappa), \quad \cot\kappa = \cot A + \cot B + \cot C,$$

$$\operatorname{cosec}\kappa^2 = \operatorname{cosec} A^2 + \operatorname{cosec} B^2 + \operatorname{cosec} C^2. \quad \text{(Simmons, Art 13, p. 106.)}$$

In Section 14, he shows that the angles about M are equal to the supplements of the angles of the triangle ABC (Simmons, Art. 8, p. 104), and proves the proportion $BD : DC = c^2 : a^2$ (Simmons, Ex. 2, p. 107), Ex. 2), if D be the point of intersection of AM and BC.

In Section 15, he deduces the equation $\cot\kappa = \dfrac{a^2+b^2+c^2}{4\Delta}$, (Simmons, Art. 15, p. 107).

In Section 16, Crelle constructs the angle κ as angle R of a triangle PQR, in which $PQ = a\sin B$, $QR = c\sin C$, $\angle PQR = \pi - C$.

In Section 29, the author considers Angular Transversals which with the adjacent sides of the triangle ABC in positive order form equal angles K, but do not pass through one point (Simmons, p. 178, Ex. 44—53).

In Section 32, he shows that the triangle Δ' embraced by the angular transversals is similar to the original triangle Δ, and that corresponding lines of the two triangles stand in the relationship $\sin(K-\kappa) : \sin\kappa$. He recognises that the maximum of Δ' takes place for $K = \kappa \pm \frac{1}{2}\pi$, and that $\Delta' = \Delta'_{\max.} - \Delta$, if the sides of Δ' are perpendicular to those of Δ.

In Section 34, finally, the author proves the proposition which, in our present terminology, runs thus :—'The positive Brocard-point of a triangle ABC, and that of a circumscribed triangle DEF whose sides form equal angles with those of ABC in positive order, fall together.' (Simmons, Art. 26, p. 116.)

These are the discoveries in the Recent Geometry of the Triangle which we owe to Crelle; and they are the oldest that are known to me.

It seems surprising (Section 14) that the simple construction of the point Ω (SIMMONS, Art. 7, p. 104) escaped CRELLE. This, however, was soon after given by C. F. A. JACOBI in his paper 'De Triangulorum Rectilineorum proprietatibus quibusdam nondum satis cognitis' (Numburgi, 1825).

The little pamphlet of CRELLE, from which the foregoing propositions relative to Recent Geometry are taken, forms the starting-point for the investigations on the Brocard-points, due to JACOBI, and afterwards to WIEGAND (1854), EMSMANN (1854), and HELLWIG (1855).

C. F. A. JACOBI, Professor of Mathematics and Physics at the Landesschule Pforta, near Naumburg an der Saale, proceeds in the above-mentioned paper from CRELLE's definition of the point Ω (modern terminology and signification are throughout used in the following), and gives in § 20 the well-known construction for that point. In § 21, the proposition is proved: If perpendiculars ΩD, ΩE, ΩF be drawn to the sides of the triangle ABC, the triangle DEF is similar to ABC and has the same first Brocard-point. (Compare SIMMONS, Art. 27.) [This may be considered as a special case of the inversion of CRELLE's theorem (CRELLE, Section 34).] In § 22, JACOBI deduces the formula

$$\cot\omega = \frac{a^2+b^2+c^2}{4\Delta}$$

in the following simple manner. Denote BD, CE, AF by a_1, b_1, c_1 respectively. Now $a_1{}^2+b_1{}^2+c_1{}^2 = (a-a_1)^2+(b-b_1)^2+(c-c_1)^2$;

$$\therefore\ aa_1+bb_1+cc_1 = \tfrac{1}{2}(a^2+b^2+c^2); \qquad \because\ a_1 = \Omega\text{D}.\cot\omega, \ldots,$$

$$\therefore\ (a.\Omega\text{D}+b.\Omega\text{E}+c.\Omega\text{F})\cot\omega = \tfrac{1}{2}(a^2+b^2+c^2);$$

$$2\Delta\cot\omega = \tfrac{1}{2}(a^2+b^2+c^2).$$

In § 23, the formula due to CRELLE, DEF $= \Delta \sin\omega^2$ (SIMMONS, p. 173, Ex. 2) is proved in a simple manner.

In §§ 24, 25, relations are given between the 18 lines produced on the sides and the angular transversals $\text{A}\Omega_a$, $\text{B}\Omega_b$, $\text{C}\Omega_c$ passing through Ω, *e.g.*,

$$\text{A}\Omega_a.\text{B}\Omega_b.\text{C}\Omega_c : \text{B}\Omega_a.\text{C}\Omega_b.\text{A}\Omega_c = \text{C}\Omega_a.\text{A}\Omega_b.\text{B}\Omega_c : \Omega\Omega_a.\Omega\Omega_b.\Omega\Omega_c \ldots (1),$$

$$\text{A}\Omega.\text{B}\Omega.\text{C}\Omega : \Omega\Omega_a.\Omega\Omega_b.\Omega\Omega_c = \text{BC}.\text{CA}.\text{AB} : \text{B}\Omega_a.\text{C}\Omega_b.\text{A}\Omega_c \ldots (2).$$

The latter relation is transformed by aid of CRELLE's formulæ in

$$\text{A}\Omega.\text{B}\Omega.\text{C}\Omega : \Omega\Omega_a.\Omega\Omega_b.\Omega\Omega_c = (b^2+c^2)(c^2+a^2)(a^2+b^2) : a^2b^2c^2.$$

(The formula (2) holds in general for any three concurrent angular transversals.)

§ 26. "If AΩ, BΩ, CΩ be produced to meet the circumcircle again in α, β, γ, and if Aβ, βC, Cα, αB, Bγ, γA be joined, the six triangles round about the point Ω are similar, and in the hexagon AβCαBγ the products of the alternate sides are equal one to another."

In § 27, the author gives the construction of the second Brocard-point, the formula for the angle $\Omega'\text{AC} = \omega'$, and for the segments of the sides formed by producing AΩ', BΩ', CΩ'.

§ 28. From $\cot\omega' = \cot\omega$ is deduced $\omega' = \omega$.

$$\text{A}\Omega.\text{B}\Omega.\text{C}\Omega = \text{A}\Omega'.\text{B}\Omega'.\text{C}\Omega' \text{ (SIMMONS, p. 107, Ex. 1.)}$$

$$\text{A}\Omega_a.\text{B}\Omega_b.\text{C}\Omega_c = \text{A}\Omega'_a.\text{B}\Omega'_b.\text{C}\Omega'_c,$$

$$\Omega\Omega_a.\Omega\Omega^b.\Omega\Omega_c = \Omega'\Omega'_a.\Omega'\Omega'_b.\Omega'\Omega'_c,$$

$$\text{B}\Omega_a.\text{C}\Omega_b.\text{A}\Omega_c = \text{C}\Omega'_a.\text{A}\Omega'_b.\text{B}\Omega'_c,$$

$$\text{A}\Omega : \text{A}\Omega' = b : c.$$

§ 30. "If BΩ, CΩ' intersect in A$'$, CΩ, AΩ' in B$'$, AΩ, BΩ' in C$'$, the sum of the isosceles triangles BCA$'$, CAB$'$, ABC$'$ is equal to the triangle ABC."

The §§ 31 to 37 are spent in investigations on the Gergonne-point, which had been treated a short time ago, in 1822, by FEUERBACH in his little book: "Properties of some remarkable Points of the Triangle" (Nürnberg). In § 38 the Nagel-point and its relation to the Gergonne-point are mentioned.

§ 39. "If AI, AK, BK, BH, CH, CI be isogonal conjugates of a triangle ABC, the lines AH, BI, CK are concurrent." (This proposition affords a simple proof of the fact that a triangle and its first Brocard triangle are collinear.)

The foregoing are the results of JACOBI referred to the Recent Geometry of the Triangle as far as they are given in the first section: "De lineis transversis in uno puncto invicem sese transeuntibus."

2. The subjoined notes on the Brocard-points (see *Companion*, p. 183) are from a letter by Professor WOLSTENHOLME:—

"In the Cambridge Math. Tripos Examination, in January, 1871, the following question was set by Mr. R. K. MILLER. 'Three particles A, B, C start from rest and move with uniform velocities, A always directing its course towards B, B towards C, and C towards A. Prove that, if their velocities be proportional to b^2c, c^2a, a^2b, where a, b, c are the initial distances of B from C, C from A, and A from B respectively, they will describe similar equiangular spirals with a common pole.' The common pole of the three spirals is obviously one of the Brocard-points. From work suggested to me by this question, I set the following in June, 1871, in a Problem Paper common to Jesus, Christ's, and Emmanuel Colleges: 'A triangle ABC is circumscribed about a fixed ellipse, focus S, such that the angles SBC, SCA, SAB are all equal; prove that each of them $= \sin^{-1}(b/2a)$, and that the angular points of the triangle lie on one of two fixed circles whose radius is $(2a^2/b)$; $2a$, $2b$ being the axes of the ellipse.' This question was sent by me about the same time to the Mathematical Editor of the *Educational Times*. It appeared there as a question for solution soon after, and solutions by myself and Mr. TUCKER may be seen in Vol. XXII. (for 1875), page 58. In these solutions are given several of the results afterwards known as Brocardian. A somewhat fuller statement was made by me in Question 1056 of the second edition of my book of problems [the facts that $\tan\theta = \sin A \sin B \sin C / (1 + \cos A \cos B \cos C)$, and that the point of contact of BC is on the symmedian through A are added]; but there is nothing there which is not proved in the solution of mine already referred to."

3. Mr. SIMMONS, who has of late added so much to the Brocardian Geometry, sends the following notes:—

"The above remarks of Prof. EMMERICH throw a flood of new light on the early history of the subject. It is curious to note how the two expressions for $\cot\omega$ and $\operatorname{cosec}^2\omega$ have been gradually traced further into the past. The latter was once thought to be due to Prof. NEUBERG (see Mr. TUCKER's paper on the T. R. Circle in the *Q. Journal*, Vol. XIX., No. 76, p. 348). Mr. M'CAY subsequently (*R. I. Academy*, July 1885) remarked that M. BROCARD had traced back many of the properties of ω to the *Problèmes de Trigonométrie* of LÉONCE CLARKE, 1849; adding that the two formulæ in question occur in HIND'S *Trigonometry*, 1841. M.

LEMOINE, at Grenoble in 1885, referred them further back to a Dutch work published in 1833 (see *Companion*, p. 182), and now Prof. EMMERICH takes a still bolder leap into antiquity, to the year 1816. It is now more than ever remarkable that the *importance* of the Brocard-points lay unrecognised until M. BROCARD himself, unaware of previous researches, again drew attention to them.

I should like to mention one or two corrections to the Historical Note (*Companion*, pp. 180—184) which were sent too late to the printers. M. LEMOINE, with whom I was discussing the proof sheets, wished me to add that his researches were given, not only at Lyons in 1873, but afterwards in a more emphatic and extended form at Lille in 1874. He also remarked that the term *Symmedian point* was invented, not by M. D'OCAGNE, but by Mr. TUCKER, the former having invented the term *Symmedian* as applied to lines. As a further insignificant instance of the difficulty of being sure of the priority of any mathematical discovery, I may perhaps add that M. LEMOINE on this occasion produced from a locker some old manuscripts, his youthful unpublished researches, long anterior to 1873. Among these I was startled to find a certain greatly cherished theorem connected with the orthocentre and circumcentre of a triangle (*Reprint*, Vol. xlviii., p. 110, and *Companion*, p. 161, Ex. 4). It is not comforting to find oneself thus anticipated by some 20 years, even though, as M. LEMOINE remarked, "cela arrive à tout le monde."

9177. (S. TEBAY, B.A.)—A straight line (ρ), drawn from the vertex of a tetrahedron to the base, makes an angle ϕ with each of the conterminous edges a, b, c; if α, β, γ be the angles included by bc, ca, ab, show that

$$3V/\rho = bc\sin\tfrac{1}{2}\alpha\,(\cos^2\tfrac{1}{2}\alpha-\cos^2\phi)^{\frac{1}{2}}+ca\sin\tfrac{1}{2}\beta\,(\cos^2\tfrac{1}{2}\beta-\cos^2\phi)^{\frac{1}{2}}$$
$$+ab\sin\tfrac{1}{2}\gamma\,(\cos^2\tfrac{1}{2}\gamma-\cos^2\phi)^{\frac{1}{2}};$$

ϕ being given by the equation $V\tan\phi = \frac{4}{3}abc\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma$.

Solution by the PROPOSER; Professor BEYENS; *and others.*

If ρ makes angles ϕ, χ, ψ with a, b, c, it has been shown that

$$\sin^2\alpha\cos^2\phi+\sin^2\beta\cos^2\chi+\sin^2\gamma\cos^2\psi-2\cos\chi\cos\psi\,(\cos\alpha-\cos\beta\cos\gamma)$$
$$-2\cos\psi\cos\phi\,(\cos\beta-\cos\gamma\cos\alpha)-2\cos\phi\cos\chi\,(\cos\gamma-\cos\alpha\cos\beta)$$
$$=1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma = m^2 \text{ (suppose)}.$$

If $\phi=\chi=\psi$, this becomes

$$m^2\sec^2\phi = \sin^2\alpha+\sin^2\beta+\sin^2\gamma-2(\cos\alpha-\cos\beta\cos\gamma)$$
$$-2(\cos\beta-\cos\gamma\cos\alpha)-2(\cos\gamma-\cos\alpha\cos\beta)$$
$$=16\sin^2\tfrac{1}{2}\alpha\sin^2\tfrac{1}{2}\beta\sin^2\tfrac{1}{2}\gamma+m^2.$$

Therefore $V\tan\phi = \frac{4}{3}abc\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma$.

Again, $V=\frac{1}{6}abc\,(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}}$;

and, changing the notation, $\text{Vol.}\,(bc\rho) = \frac{1}{3}bc\rho\sin\frac{1}{2}\alpha\,(\cos^2\frac{1}{2}\alpha-\cos^2\phi)^{\frac{1}{2}}$,

$$\text{Vol.}\,(ca\rho) = \tfrac{1}{3}ca\rho\sin\tfrac{1}{2}\beta\,(\cos^2\tfrac{1}{2}\beta-\cos^2\phi)^{\frac{1}{2}},$$
$$\text{Vol.}\,(ab\rho) = \tfrac{1}{3}ab\rho\sin\tfrac{1}{2}\gamma\,(\cos^2\tfrac{1}{2}\gamma-\cos^2\phi)^{\frac{1}{2}}.$$

And, since $V=\text{Vol.}\,(bc\rho)+\text{Vol.}\,(ca\rho)+\text{Vol.}\,(ab\rho)$, therefore, &c.

9192 (G. G. MORRICE, M.A.)—Take three points on a sphere

$$z_1,\ z_2 = \frac{(d+ic)\,z_1-(b-ia)}{(b+ia)\,z_1+(d-ic)},\quad z_3 = \frac{(d'+ic')\,z_2-(b'-ia')}{(b'+ia')\,z_2+(d'-ic')}$$

$$= \frac{(d''+ic'')\,z_1-(b''-ia'')}{(b''+ia'')\,z_1+(d''-ic'')},$$

z_1 denoting the complex variable x_1+iy_1; and show that the poles of the three great circles joining these points are connected by the linear transformations

$$Z_3 = \frac{(\delta''+i\gamma'')\,Z_1-(\beta''-i\alpha'')}{(\beta''+i\alpha'')\,Z_1+(\delta''-i\gamma'')},\ \&c.,$$

where

$$\frac{\alpha''}{b'c-bc'} = \dots \frac{\delta''}{\sqrt{1-d^2\,.\,1-d'^2}-aa'-bb'-cc'}.$$

Solution by the PROPOSER; Professor MATZ, M.A.; *and others.*

If $\xi_1\eta_1\zeta_1$, $\xi_2\eta_2\zeta_2$, $\xi_3\eta_3\zeta_3$ denote the Cartesian coordinates of the three points, $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$, &c., those of the poles Z_1, Z_2, Z_3; we have

$$(1)\quad a'' = \bar{\xi}_2 \sin\frac{z_1z_3}{z},\quad b'' = \bar{\eta}_2 \sin\frac{z_1z_3}{z},\quad c'' = \bar{\zeta}_2 \sin\frac{z_1z_3}{z},\quad d'' = \cos\frac{z_1z_3}{z};$$

$$(2)\quad \alpha'' = \xi_2 \cos\frac{z_2}{z},\quad \beta'' = \eta_2\cos\frac{z_2}{z},\quad \gamma'' = \zeta_2\cos\frac{z_2}{z},\quad \delta'' = \sin\frac{z_2}{z};$$

where in (1) we suppose the transformation from one point to another to take place by a rotation round the central axis through the corresponding pole, and in (2) we suppose the same generation of the polar triangle.

Now we have,

$$\frac{\xi_2}{\bar{\eta}_1\bar{\zeta}_3-\bar{\eta}_3\bar{\zeta}_1} = \frac{\eta_2}{\dots} = \frac{\zeta_2}{\dots} = \frac{1}{\sin z_2};$$

hence, substituting, we obtain

$$\frac{\alpha''}{b'c-bc'} = \dots = \frac{\delta''}{\sqrt{1-d^2\,.\,1-d'^2}-aa'-bb'-cc'}.$$

9540 (Professor IGNACIO BEYENS.)—Si t_a, t_b, t_c sont les tangentes menées des sommets d'un triangle ABC au cercle des neuf points, et S la surface du triangle ABC, $S = (t_a^2t_b^2+t_a^2t_c^2+t_b^2t_c^2)^{\frac{1}{2}}$.

Solution by J. O'BYRNE CROKE, M.A., *and* R. KNOWLES, B.A.

Let D, E, F,—G, H, I,—be respectively the first three, the feet of the perpendiculars from the opposite angles on the sides a, b, c of the triangle ABC, and the second three, the middle points of these sides. Then

$$\Sigma\,(t_a^2t_b^2) = \Sigma\,(\text{AI}\,.\,\text{AF}\,.\,\text{BG}\,.\,\text{BD}) = \Sigma\,(\tfrac{1}{4}\,abc\,c\cos\text{A}\cos\text{B})$$

$$= \tfrac{1}{8}\,\Sigma\left[(b^2+c^2-a^2)\,(a^2+c^2-b^2)\right]$$

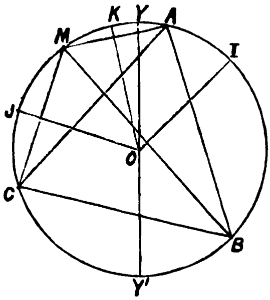
$$= \tfrac{1}{16}\,(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4) = \tfrac{1}{4}\,b^2c^2\sin^2\text{A} = \text{S}^2;$$

hence we have

$$\text{S} = (t_a^2t_b^2+t_b^2t_c^2+t_c^2t_a^2)^{\frac{1}{2}}.$$

9576. (Professor Mannheim.)—On donne une droite arbitraire, une circonférence et un point M sur cette courbe. Mener de ce point une corde MA telle que la tangente en A à la circonférence et cette corde soient également inclinées sur la droite donnée. Il existe trois cordes telles que MA; si A, B, C désignent leurs extrémités, démontrer que les tangentes en ces points forment un triangle équilatéral.

Solution by Prince Camille de Polignac.



For convenience we may substitute in data—for the arbitrary line, the diameter YY′ perpendicular to its direction; for the chord MA and the tangent in A respectively, the radius OK perpendicular to MA and the normal OA. Supposing, which is allowed, arc MY < $\frac{1}{2}\pi$, we take a point A on the other side of Y, such that arc AY = $\frac{1}{3}$ arc YM, and draw the equilateral triangle ABC. Its summits will be the sought points, as appears from the following proof.

For A.—The radius OK, perpendicular to MA, and the normal OA, make, by the construction, equal angles with diameter YY′, as should be the case according to the modified data.

For B.—Draw radius OI perpendicular to MB; we have to show that arc YI = arc BY′. Put arc YA = α; then we have

$$\text{arc MAB} = 4\alpha + \tfrac{2}{3}\pi,\ \ \tfrac{1}{2}\text{MAB} = 2\alpha + \tfrac{1}{3}\pi = \text{MI},\ \ \text{MI} - 3\alpha = \text{YI} = \tfrac{1}{3}\pi - \alpha,$$
$$\text{arc BY}' = \pi - \alpha - \tfrac{2}{3}\pi - \tfrac{1}{3}\pi - \alpha.$$

For C.—Draw radius OJ perpendicular to MC; we have to show that arc YJ = arc CY′. Now,

$$\text{MC} = \text{AC} - \text{AM} = \tfrac{2}{3}\pi - 4\alpha,\ \ \tfrac{1}{2}\text{MC} = \tfrac{1}{3}\pi - 2\alpha = \text{MJ},\ \ \text{MJ} + 3\alpha = \tfrac{1}{3}\pi + \alpha = \text{YJ};$$

on the other hand, $\text{CY}' = \pi - \text{CMA} + \text{AY} = \pi - \tfrac{2}{3}\pi + \alpha = \tfrac{1}{3}\pi + \alpha.$

There can be no other point on semicircle YAY′ answering the question, as when BY′ decreases YI increases and *vice versa*. Same remark for the other semicircle. The tangents in A, B, C obviously form an equilateral triangle.

9571. (Professor Sylvester, F.R.S.)—Let $f\theta$ represent any finite rational integer function of θ with integer coefficients, and $u_{x+1} = fu_x$, and $u_1 = f0$; show that, if δ is the greatest common measure of r, s, then u_δ will be the greatest common measure of u_r, u_s.

Solution by W. S. Foster, M.A.

Let $f(\theta) = a(\theta+\beta)(\theta+\gamma)\dots(\theta+\mu)$, therefore $u_1 = a\beta\gamma\dots\mu$,

$u_2 = a(u_1+\beta)(u_1+\gamma)\dots(u_1+\mu)$, $u_3 = a(u_2+\beta)(u_2+\gamma)\dots(u_2+\mu)$, and so on.

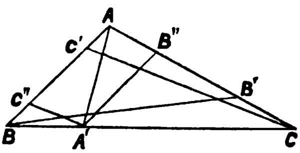
Then $\qquad u_r = \mathrm{M}(u_s) + u_{r-s} =$ also $\mathrm{M}'(u_{r-s}) + u_s$;

therefore the G. C. M. of u_r and u_s = G. C. M. of u_r and u_{r-s}, and u_{2s}, u_{3s} ... are all multiples of u_s.

If δ be the G. C. M. of r and s, let $pr-qs = \delta$, therefore G. C. M. of u_{pr}, u_{qs} = G. C. M. of u_{pr} and u_δ, $= u_\delta$; therefore G. C. M. of u_r and u_s cannot be greater than u_δ, and we know that u is a common measure of u_r and u_s; hence u_δ is the greatest common measure of u_r and u_s.

9583. (Professor Neuberg.)—On divise les côtés d'un triangle ABC aux points A′, B′, C′ en parties proportionnelles, de manière que BA′/A′C = CB′/B′A = AC′/C′B. Démontrer que les forces représentées par les droites AA′, BB′, CC′ se réduisent à un couple, et conclure de là que ces droites représentent en grandeur et en direction les côtés d'un triangle.

Solution by Professor De Wachter; J. Young, M.A.; *and others.*



If $\mathrm{AB} = \alpha$, $\mathrm{BC} = \beta$, $\mathrm{CA} = \gamma$, and the common ratio

$$\mathrm{AC}' : \mathrm{C'B} = \mathrm{BA}' : \mathrm{A'C} = \mathrm{CB}' : \mathrm{B'A} = k,$$

then we have $\alpha + \beta + \gamma = 0$,

$$\mathrm{AA}' = \alpha + k\beta, \quad \mathrm{BB}' = \beta + k\gamma, \quad \mathrm{CC}' = \gamma + k\alpha;$$

hence $\qquad \mathrm{AA}' + \mathrm{BB}' + \mathrm{CC}' = (\alpha+\beta+\gamma)(1+k) = 0.$

Thus, since the vectors AA′, BB′, CC′ are the sides of a certain triangle, by means of a parallel translation, they may also represent, in the same translated position, a system of forces in equilibrium, and, consequently, a system of forces reducible to a couple, when taken as in the figure.

[If we draw A′B″, A′C″ parallel to AB, AC, then AC is divided in the same ratio at B′, B″; thus AB″ = CB′, and, in like manner, AC′ = BC″. Now, the force AA′ may be resolved into the forces AC″, AB″ or C′B, B′C; thus, the three forces in question become C′B, B′C, A′C, C′A, B′A, A′B, acting in the directions indicated by the letters; and these six, taken in pairs A′C, A′B, &c., reduce to three forces, one along each side of the triangle, and proportional to the difference of the segments. As the difference of the segments is proportional to their sum, the rest of the proof is obvious.]

2505. (Rev. R. H. Wright, M.A.)—Prove that

$$\frac{(1+\frac{1}{2}x)^2}{(1+\frac{1}{3}x)(1+\frac{1}{3}x)} \cdot \frac{(1+\frac{1}{4}x)^2}{(1+\frac{1}{5}x)(1+\frac{1}{5}x)} \text{ ad inf.} = \frac{2^{-x}\,\Gamma(2+x)}{\Gamma(\frac{1}{2}x+1)\left\{\Gamma(\frac{1}{2}x+1)\right\}^2}.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

We have $$\log\Gamma(1+x) = -Cx + \frac{S_2x^2}{2} - \frac{S_3x^3}{3} + \dots,$$

where $S_r = 1+\frac{1}{2^r}+\frac{1}{3^r}+\dots$, also $\frac{1}{3^r}+\frac{1}{5^r}+\dots = S_r - 1 - \frac{S_r}{2^r}$,

Hence taking logarithms

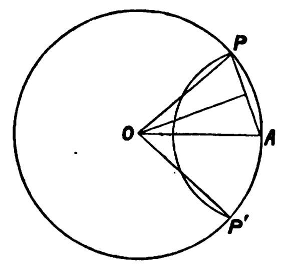
$$2\left\{\left(\frac14+\frac18+\dots\right)x - \frac{S_2x^2}{2\,.\,4^2}+\frac{S_3x^3}{3\,.\,4^3}-\dots\right\}$$
$$-\left(\frac13+\frac15+\dots\right)x + \left(\frac1{3^2}+\frac1{5^2}+\dots\right)\frac{x^2}{2} - \left(\frac1{3^3}+\frac1{5^3}+\dots\right)\frac{x^3}{3}+\dots$$
$$= (1-\log 2)\,x - \frac{2S_2x^2}{2\,.\,4^2}+\frac{2S_3x^3}{3\,.\,4^3}-\dots+\frac{S_2x^2}{2}-\frac{S_3x^3}{3}+\dots$$
$$-\frac{S_2x^2}{2\,.\,2^2}+\frac{S_3x^3}{3\,.\,2^3}-\dots-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$
$$= -x\log 2 - 2\log\Gamma(\tfrac14 x+1) - \tfrac12 Cx + \log\Gamma(1+x)$$
$$+ Cx - \log\Gamma(\tfrac12 x+1) - \tfrac12 Cx + \log(1+x)$$
$$= \log 2^{-x} + \log\Gamma(2+x) - \log[\Gamma(\tfrac14 x+1)]^2 - \log\Gamma(\tfrac12 x+1),$$

since $$(1+x)\,\Gamma(1+x) = \Gamma(2+x),$$

and the given product is equal to the result stated.

3499. (W. C. OTTER, F.R.A.S.)—A gentleman has a circular meadow whose area is just five acres, surrounded by an iron palisading, to the inside of which he wishes to tether his horse so as to enable him to graze over just *one* acre of ground. Find, by a general method, what must be the length of the tether.

Solution by D. BIDDLE.



Let O be the centre of the field; A, the fastened end of rope; AP, the required length of tether; a, the proportion borne to the whole area by that over which the horse may graze (in this instance, $\frac15$); and $\phi = \angle AOP$. Hence

$$\phi - \sin\phi + 4\sin^2\tfrac12\phi\,(\tfrac12\pi - \tfrac12\phi) = a\pi.$$

But $$2\sin^2\tfrac12\phi = 1-\cos\phi,$$

$$\therefore\ \sin\phi + (\pi-\phi)\cos\phi = (1-a)\,\pi\ \dots\dots(1).$$

In the present case, it is easy to see that ϕ cannot be far from 40°. But, on trying it, we find $\cdot6427876 + 2\cdot4434609 \times \cdot7660444 = 2\cdot5132741 + \cdot0013130$, the second term on the right being the amount of error.

In order to rectify the error, render this equation

$$n+pq = m+e, \text{ whilst } (n+y)+(p-x)(q-z) = m \ldots\ldots\ldots\ldots (2, 3).$$

By referring to the tables of natural sines, cosines, and arcs, we find that at 40°, whilst ϕ varies $+\cdot 0002909$, $\cos\phi$ varies $-\cdot 0001870$, and $\sin\phi$ varies $+\cdot 0002228$. Roughly, therefore, $y = kx$, and $z = lx$, where $k = \cdot 7658989$, $l = \cdot 6428326$. And by subtracting (2) from (3) on substituting these values, we have a quadratic which yields

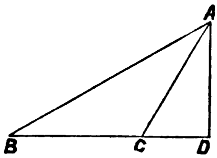
$$x = \pm\left\{\left(\frac{pl+q-k}{2l}\right)^2 - e\right\}^{\frac{1}{2}} + \frac{pl+q-k}{2l} = \cdot 0005381, \text{ as one root.}$$

Therefore $\quad \phi = \cdot 6986698 = 40^\circ\, 1'\, 51''$, and $AP/AO = \cdot 6845460$.

If the first error had not been slight, more than one correction would have been needed.

9596. (Rev. T. C. Simmons, M.A.)—Prove *geometrically* that, when two angles of a triangle differ by 90°, the centre of the nine-point circle lies on the intercepted side.

Solution by the Proposer; Professor Emmerich; *and others.*



Let ABC be a triangle in which $\angle C = 90^\circ + B$; draw AD perpendicular on BC. Then $BCA = 90^\circ + CAD$, whence $\angle ABC = \angle CAD$, and $DB \cdot DC = DA^2$. Therefore DA touches the circumcircle of ABC. But DA passes through the orthocentre of ABC, which is a centre of similitude of its circumcircle and nine-point circle. Therefore DA touches the latter circle also, and D is the point of contact; therefore the nine-point centre lies on BC.

[The angle between the perpendicular from A to the base BC and the line joining A to the circumcentre V, being equal to the difference of the angles B and C, when B and C differ by 90°, the circumcentre lies on a parallel through A to BC. Applying this fact to the triangle whose sides join the mid-points of the original triangle, the proposition is proved.]

9234. (Professor Mahendra Nath Ray, M.A., LL.B.)—Show that the sum to infinity of the series

$$\frac{1 \cdot 2x}{3 \cdot 4} - \frac{2 \cdot 3x^2}{4 \cdot 5} + \frac{3 \cdot 4x^3}{5 \cdot 6} - \ldots\ldots \text{ is } \left(\frac{3}{x^3} + \frac{1}{x^2}\right)\log(1+x)^2 - \frac{5x+6}{x^2(1+x)}.$$

Solution by R. Knowles, B.A.; Professor Sarkar, M.A.; *and others.*

Using Boole's symbolical method, $\phi(m) = \dfrac{m(m+1)}{(m+2)(m+3)}$, therefore

$$u = \frac{D(D+1)}{(D+2)(D+3)} \cdot \frac{\epsilon^\theta}{1+\epsilon^\theta} = \left[1 + 2(D+2)^{-1} - 6(D+3)^{-1}\right]\frac{\epsilon^\theta}{1+\epsilon^\theta}$$

$$= \frac{\epsilon^\theta}{1+\epsilon^\theta} + 2\epsilon^{-2\theta}\int\left(\epsilon^{2\theta} - \epsilon^\theta + \frac{\epsilon^\theta}{1+\epsilon^\theta}\right)d\theta - 6\epsilon^{-3\theta}\int\left(\epsilon^{3\theta} - \epsilon^{2\theta} + \epsilon^\theta - \frac{\epsilon^\theta}{1+\epsilon^\theta}\right)d\theta,$$

and performing the integrations and putting $x = \epsilon^\theta$, we have the result.

9549. (The EDITOR.)—Through two given points (A, B) draw a circle such that its chord of intersection with a given circle may pass through a given point (C).

Solution by W. S. FOSTER; Professor ABINASH CHANDRA BASU; *and others.*

Draw CT a tangent to the given circle; take on CA a point P such that PC . CA = CT². Then the circle drawn through the points P, A, B will be the circle required; for the tangents from C to the two circles are equal, hence their chord of intersection will pass through C.

[The diagram, being easily imagined, is unnecessary.]

9569. (ASPARAGUS.)—Prove that the curve whose equation is

$$(y^2+28ax+96a^2)^2 = 64a\,(x+3a)(x+7a)^2$$

is *unicursal*, of the sixth class, has three acnodes, three axial foci (one coinciding with a node), one bi-tangent (contacts impossible), and two inflexions, at each of which the tangent has four-point contact.

Solution by Professor NASH, M.A.

The right-hand side of the equation is a square if $x+3a = a\theta^2$. This gives $\dfrac{y^2}{4a^2} = (\theta-1)^2(2\theta-3)$; therefore, if $2\theta-3 = \lambda^2$, x and y are both rational functions of λ, and the curve is unicursal. The values are

$$4x = a\,(\lambda^4+6\lambda^2-3),\quad y = a\lambda\,(1+\lambda^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\text{(A)}.$$

Hence the equation of the tangent is

$$4x\,(1+3\lambda^2)-4y\,(\lambda^3+3\lambda)+a\,(\lambda^6-3\lambda^4+15\lambda^2+3) = 0.$$

This being of the 6th degree in λ, the curve is of the 6th class.

The coordinates of the double points may be obtained from the conditions that the two equations (A) should have two roots common. Finding the G.C.M., the conditions are

$$\frac{-y}{5a} = \frac{8a+4x}{y} = \frac{5y}{3a+4x}.$$

These give $x=-2a$, $y=0$, $\lambda=\pm(-1)^{\frac{1}{2}}$, or $x=-7a$, $y=\pm 10a$, $\lambda=\pm1\pm2\,(-1)^{\frac{1}{2}}$. Substituting for x and y in the equation of the tangent their values in terms of λ', and dividing the resulting equation by $(\lambda'-\lambda)^2$, we get a quadratic to determine the other two points where any tangent meets the curve. This quadratic is

$$\lambda'^2\,(1+3\lambda^2)+2\lambda'\,(\lambda^3-5\lambda)+\lambda^4-3\lambda^2+6 = 0 \;\ldots\ldots\ldots\ldots\text{(B)}.$$

If this is satisfied by $\lambda'=\lambda$, the point λ must be a point of inflexion. Making the substitution, we get $(\lambda^2-1)^2 = 0$. Hence $\lambda=\pm1$ are points of inflexion, and the tangents at them meet the curve in four consecutive points.

The bi-tangents are found from the condition that (B) should have equal roots. This condition is $(\lambda^2-1)^2(\lambda^2+3) = 0$. Hence, besides the

inflexional tangents, there is one bi-tangent $x=-3a$, its points of contact being impossible ($y^2+12a^2=0$). The inflexional tangents are $y^2=(x+a)^2$, and the points of contact are $(a, \pm 2a)$. The equation of the curve can be written

$$y^4+2y^2(28ax+96a^2)-16a(x+2a)^2(4x+3a)=0.$$

If $x<-3a$ all the four values of y are imaginary, therefore in the real part of the curve the coefficient of y^2 is positive. The last term is negative if $x<-\frac{3}{4}a$, and positive if $x>-\frac{3}{4}a$; therefore, if $x<-\frac{3}{4}a$, there is no real value of y, and, if $x>-\frac{3}{4}a$, there are only two real values. Hence the curve consists of one infinite branch with its vertex at $(-\frac{3}{4}a, 0)$, touching the lines $\pm y=x+a$ at the points $(a \pm 2a)$. The distance between the curve and these lines increases so slowly that it is difficult to show the shape of the curve, except by drawing it on a large scale.

[The inflexional tangents are $y^2=(x+a)^2$, and the curve is remarkably near to these tangents for a considerable range of values of x, say from $x=0$ to $x=13a$. When $x=0$, the real positive value of y in the curve is $a(56\sqrt{3}-96)^{\frac{1}{2}}$, and in the tangent $y=x+a$ is a, and the difference is ($\cdot 0025a$) very small. When $x=13a$, the two ordinates are $(180)^{\frac{1}{2}}a$ and $14a$, and the difference is about $\frac{4}{7}a$.]

9594. (R. Knowles, B.A.)—From a fixed point T, on the director circle of an ellipse, tangents are drawn to the ellipse; a third tangent at a variable point R on the ellipse meets these in M, N respectively. Prove that the locus of the mid-point of MN is a rectangular hyperbola.

9599. (D. Edwardes.)—Find the equation of the locus of a point where a tangent to a conic meeting two fixed tangents is cut in a given ratio $\mu : 1$.

Solution by Emily Perrin; J. Young, M.A.; *and others.*

(9599.) If the fixed tangents be taken as axes, the conic is

$$xy=(ax+by+c)^2,$$

and a tangent to this at the point λ, where

$$x\lambda^{-1}=ax+by+c, \quad y\lambda=ax+by+c,$$

is $x-2\lambda(ax+by+c)+y\lambda^2=0$. The coordinates of the point dividing the intercept in ratio $\mu : 1$, are

$$x=\frac{2\lambda c}{(1-2a\lambda)(1+\mu)}, \quad y=\frac{2\lambda\mu c}{(\lambda^2-2b\lambda)(1+\mu)};$$

hence the locus is the conic

$$4\left[\mu c+by(1+\mu)\right]\left[c+a(1+\mu)x\right]=(1+\mu)^2 xy.$$

[If the given curve be a parabola $4ab=1$, and this conic breaks up into the line at infinity, and the finite line, $a\mu x+by+\mu c/(1+\mu)=0$.]

(9594.) This proposition is a particular case of Question 9599.

[The locus of the point dividing in *any* fixed ratio the intercept on a variable tangent will be a rectangular hyperbola, having its asymptotes parallel to the tangents from T.]

9029. (Professor BORDAGE.)—Show that the roots of the equation

$$\left[\log(x+1)\right]^2+\left[2\log 2+\log(x^2-1)\right]\log(x+1)$$
$$-\left[\log(x-1)+\log(x^2-1)\right]\log(x-1)+(\log 2)^2+\log(x^2-1)\log 2=0$$

are -3 and $\pm(1+2^{-\frac{1}{2}})^{\frac{1}{2}}$.

Solution by G. G. STORR, M.A.; FANNIE H. JACKSON; *and others.*

Here
$$\left[\log(x+1)\right]^2+\tfrac{3}{2}\log 2\log(x+1)+\tfrac{9}{16}(\log 2)^2$$
$$=\log(x-1)^2-\tfrac{1}{2}\log 2\log(x-1)+\tfrac{1}{16}(\log 2)^2;$$

therefore $\log(x+1)+\tfrac{3}{4}\log 2=\pm\left[\log(x-1)-\tfrac{1}{4}\log 2\right]$;

whence $\log(x+1)/(x-1)=\log\tfrac{1}{2}$ or $\log(x^2-1)=\log(1/\sqrt{2})$, therefore, &c.

9592. (R. W. D. CHRISTIE.)—ABC is a right-angled triangle; CD cuts the hypotenuse in D. The angle BCD $=52\tfrac{1}{2}°$. The angle DBC $=7\tfrac{1}{2}°$. Instead of these substitute two other angles without altering the ratio of BD : AD.

Solution by J. YOUNG, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Draw CD′, BD′ perpendicular to BD, CD respectively; let BD′, CA meet in A′. The angle DBD′ = DCD′ = 90° − CDA = 30°, therefore BCD′ = $82\tfrac{1}{2}°$ and D′BC = $37\tfrac{1}{2}°$. Also, since DD′ is parallel to CA, the ratio BD : DA = BD′ : D′A′.

[*Otherwise*:—Draw DE perpendicular to BC, then BD/DA = BE/EC $=\cot 7\tfrac{1}{2}°/\cot 52\tfrac{1}{2}°=\cot 37\tfrac{1}{2}°/\cot 82\tfrac{1}{2}°$; thus the given angles may be replaced by $82\tfrac{1}{2}°$ and $37\tfrac{1}{2}°$.]

9409. (F. R. J. HERVEY.)—The focal chords ASB, CSD of a rectangular hyperbola are at right angles; normals at A, B meet at P, and normals at C, D at Q. Prove that PQ is trisected by the focal chords, and bisected by the directrix corresponding to S.

Solution by R. KNOWLES, B.A.; Professor CHAKRAVARTI; *and others.*

If the coordinates of the pole of ASB be $a\sqrt{2},\ k$; then those of CSD are $a/\sqrt{2},\ -a^2/2k$; hence we find those of P and Q to be

$$2^{\frac{3}{2}}a(a^2+k^2)/(a^2-2k^2),\quad 2a^2k/(a^2-2k^2);$$
$$\sqrt{2}a(a^2+4k^2)/(2k^2-a^2),\quad 2a^2k/(a^2-2k^2)\ \text{respectively};$$

therefore PQ is parallel to the axis of the curve, and the abscissæ of the points where ASB, CSD meet PQ are $2^{\frac{3}{2}}ak^2/(2k^2-a^2)$ and $2^{\frac{1}{2}}a^3/(a^2-2k^2)$; therefore (1) PQ is trisected in these points, and (2) the abscissa of the mid-point of PQ is $2^{\frac{1}{2}}a$, and it is on the directrix.

9577. (Professor GENESE, M.A.)—Prove that similarly placed conics with a common orthocycle (director-circle) are inscribed in the same square. Also that, if two such conics be drawn through any point, the tangents at the point are equally inclined to a side of the square.

Solution by J. YOUNG, M.A.; SARAH MARKS, B.Sc.; *and others.*

If the line $lx+my=n$ touches the ellipse $b^2x^2+a^2y^2=a^2b^2$, then $l^2a^2+m^2b^2=n^2$. This condition is satisfied for all ellipses of the system by the lines $x \pm y = \pm [(a^2+b^2)]^{\frac{1}{2}}$, and these four lines evidently form a fixed square inscribed in the orthocycle.

9519. (J. BRILL, M.A.)—A particle is projected from a point on a horizontal plane with a velocity V, and in a direction making an angle α with the plane. The coefficient of elasticity between the particle and the plane is e, and the coefficient of friction is $\tan\lambda$. Prove (1) that, if $\tan\alpha < (1-e)\cot\lambda/(1+e)$, the particle will come to rest on the plane after a time $V\cos(\alpha-\lambda)/g\sin\lambda$, and at a distance $V^2\cos^2(\alpha-\lambda)/g\sin 2\lambda$ from the point of projection; and (2) find how the problem will be modified if

$$\tan\alpha > (1-e)\cot\lambda/(1+e).$$

Solution by Professor MADHAVARAO, M.A.

When the particle begins to describe the $(r+1)^{\text{th}}$ parabola, the vertical velocity is $e^rV\sin\alpha$, and the horizontal velocity is

$$V\cos\alpha-\tan\lambda\,(1+e)\,V\sin\alpha\,(1+e+e^2+\ldots+e^{r-1})$$
$$= V\cos\alpha-\tan\lambda\,(1+e)\,V\sin\alpha\,\frac{1-e^r}{1-e}.$$

(a) If $\tan\alpha<\dfrac{1-e}{1+e}\cot\lambda$, then $V\cos\alpha>\dfrac{1+e}{1-e}V\sin\alpha\tan\lambda$.

Therefore the number of rebounds will be infinite, when the vertical velocity is destroyed, and the horizontal velocity is

$$v = V\cos\alpha-\frac{1+e}{1-e}V\sin\alpha\tan\lambda,$$

with which the particle moves along the plane in contact with it.

If $T_1, T_2, T_3, \ldots$ be the times of successive rebounds,

$$T_1 = \frac{T_2}{e} = \frac{T_3}{e^2} = \ldots = \frac{2V\sin\alpha}{g},$$

therefore $T = T_1+T_2+T_3+\text{to inf.} = T_1(1+e+e^2+\ldots) = \dfrac{2V\sin\alpha}{(1-e)^2g}$.

If t = time of subsequent motion along the plane,

$$t = \frac{v}{g\tan\lambda} = \frac{V}{g}\left(\frac{\cos\alpha}{\tan\lambda}-\frac{1+e}{1-e}\sin\alpha\right),$$

therefore whole time $= T + t = \dfrac{2V\sin\alpha}{(1-e)g} + \dfrac{V}{g}\left(\dfrac{\cos\alpha}{\tan\lambda} - \dfrac{1+e}{1-e}\sin\alpha\right)$

$$= \frac{V}{g}\left(\frac{\cos\alpha}{\tan\lambda} + \sin\alpha\right) = \frac{V\cos(\alpha-\lambda)}{g\sin\lambda}.$$

Again, if R_1, R_2, R_3, ... be the ranges on the plane of successive rebounds, and s the space described in time t along the plane,

$$R_r = \frac{2e^{r-1}V\sin\alpha}{g}\left\{V\cos\alpha - \frac{1+e}{1-e}(1-e^{r-1})V\sin\alpha\tan\lambda\right\},$$

therefore $\qquad R = R_1 + R_2 + R_3 + \dots$ to inf.

$$= \frac{2V^2\sin\alpha}{(1-e)g}\left\{\cos\alpha - \frac{1+e}{1-e}\sin\alpha\tan\lambda\right\} + \frac{2V^2\sin^2\alpha\tan\lambda}{(1-e^2)g},$$

and $\qquad S = \dfrac{v^2}{2g\tan\lambda} = \dfrac{V^2}{2g\tan\lambda}\left\{\cos\alpha - \dfrac{1+e}{1-e}\sin\alpha\tan\lambda\right\}^2,$

therefore whole space $= R + S = \dfrac{V^2\cos^2(\alpha-\lambda)}{g\sin 2\lambda}.$

(2) If $\tan\alpha > \dfrac{1-e}{1+e}\cot\lambda$, then $V\cos\alpha < \dfrac{1+e}{1-e}V\sin\alpha\tan\lambda$,

therefore the particle makes n rebounds, where n is the integer satisfying the condition $0 > V\cos\alpha - \dfrac{(1+e)(1-e^n)}{1-e}V\tan\lambda\sin\alpha$

$$< V\cos\alpha - \frac{(1+e)(1-e^{n-1})}{1-e}V\tan\lambda\sin\alpha,$$

or $\qquad \tan\alpha < \dfrac{1-e}{(1+e)(1-e^n)}\cot\lambda > \dfrac{1-e}{(1+e)(1-e^{n-1})}\cot\lambda.$

Since after n rebounds, the vertical velocity is not all lost, while the horizontal velocity is destroyed, the particle will move up and down in the same vertical line, and impinge at the same point in the plane an infinite number of times.

9355. (Professor Nash, M.A.)—Supposing a chess-board to be in the form of a rectangle containing mn squares, show that a knight's tour is possible for every value of m and n (72), except when $m = 3$ and $n = 3$, 5, or 6, or $m = 4$ and $n = 4$.

Solution by the Proposer.

In order to prove that a "knight's tour" is possible on a chess-board in a form of a rectangle containing mn squares, whatever the values of m and n may be, it is sufficient to construct a certain number of tours for small values of m and n, provided that these are of such a nature that, when any two of them are placed together, the initial square of one and the terminal square of the other are a knight's move apart. This will be the case if the initial and terminal squares of each tour are respectively a corner of the rectangle, and a square diagonally adjoining another corner.

Such a tour can be constructed for all values of m and n from 5 to 9 inclusive. If m and n are both odd, the terminal square may be the square diagonally next to any of the four corners; if one of the two is odd, the terminal may be next to the corner opposite to the initial square, or the other corner on the side containing an odd number of squares. If m and n are both even, the terminal may be next to either of the corners on the diagonal which does not pass through the initial point.

Hence, whatever be the values of m and n between the given limits, the terminal point is such that a tour of the form $m \times n'$ or $m' \times n$ can be applied to any of the sides of the first rectangle, thus giving a tour $m \times (n+n')$ or $n \times (m+m')$, in which the initial and terminal squares have the same relative position as in the component tours. Hence any tour of the form $m \times \mathrm{N}$ can be obtained, where m is between the limits 5 and 9, and N any number, and the terminal square will be next to one of the corners on the side of length m which does not contain the initial point.

A knight's move from this point will bring the knight into position for commencing a fresh tour, $m' \times \mathrm{N}$ adjoining the first one, thus giving a tour $(m+m') \times \mathrm{N}$.

If either m' or N is odd, this process can be repeated indefinitely, thus giving a complete tour of the form $\mathrm{M} \times \mathrm{N}$, whatever M and N may be.

If both m' and N are even the terminal square will be in the corner adjoining the first rectangle $m \times n$, or the other corner on the same diagonal of the last elementary rectangle in the tour $m' \times \mathrm{N}$. In the first case the tour cannot be continued in this direction; in the second it can be continued, but the position of the point does not give perfect freedom of choice about the form of the next rectangle.

This difficulty can always be avoided either by taking m, m', &c. all odd, or only the first even and the rest odd, or the n of the first rectangle can be made odd and other rectangles of the form $n \times \mu$ can be applied to it, so as to complete the rectangle $n \times \mathrm{M}$, thus reducing the N in other component rectangles to an odd number. The latter method must be adopted when a re-entrant tour is required; in this case m, m', &c., must be so chosen that the number of them is even. This is always possible when M and N are both greater than 9 (except when both are 19), and therefore a re-entrant tour can always be obtained when the number of squares is even, and neither side less than 10.

[These statements are illustrated by a diagram giving a re-entrant tour for 40×22 squares, wherein, however, the tour is not constructed in the simplest manner, because Professor Nash has tried to introduce into it all the rectangles for which a tour is possible up to the square of 9. It contains all but two, 8×4 and 9×3; two others (9×8 and 8×8) are not shown explicitly, but they can be obtained by combining other rectangles (viz., 8×6 and 8×3, 8×5 and 8×3). This method of proof does not apply to rectangles of the forms $4 \times n$, $3 \times n$. It is impossible in any case to form tours for rectangles of the form $4 \times n$ by combining tours over smaller rectangles; hence the method given above cannot be applied to this case. Diagrams for 4×8 and 4×9 are given, in both of which the initial square is the fourth from the end on one of the long sides, and the terminal is the sixth from the same end on the opposite side. The first few moves are the same in both cases, and I believe the same is true for every value of n greater than 6. I have tried all values up to 18, and also 50 and 51. It is clear that two tours of this form can be placed so as to form one tour $8 \times n$, and therefore a tour of

the form $4m \times n$. Another diagram gives a re-entrant tour for 14×3 squares, which by division along two lines (AB, CD) gives open tours for 10×3, 7×3, and 4×3 squares. By omitting the move marked EF a tour 4×3 can be inserted, as shown in a smaller diagram, and this can be repeated, which gives tours for $3 \times (4n+3)$ and $3 \times (4n+6)$ squares for every value of n excluding zero. A further diagram gives a tour for 3×9 squares, and by the same method a tour for $3 \times (4n+9)$ squares can be obtained. Hence a tour of the form $3 \times n$ can be obtained for every value of n except 3, 5 and 6. Of the tours 4×4, 4×5, 4×6, the last two are shown in the large diagram, and the first is impossible. We regret that these diagrams are altogether too large for our very inadequate space, but the solution is clear enough without them.]

9527. (A. Russell, B.A.)—If a polygon be drawn round a circle prove that the area of the polygon is $S^2/(\Sigma \cot \frac{1}{2}A)$, where S denotes, half the sum of the sides, and A an angle of the polygon.

Solution by Rev. J. L. Kitchin, M.A.; Prof. Chakravarti; *and others.*

Let R be the radius of the circle, then area $= S \,.\, R$; also let the sides of the polygon be $a_1, a_2, \dots a_n, \dots$, then we have $a_1 = R(\cot \frac{1}{2}A_1 + \cot \frac{1}{2}A_2)$, therefore $S/R = \Sigma \cot \frac{1}{2}A$; and it thus follows that the area is $S^2/\Sigma \cot \frac{1}{2}A$.

9281. (S. Tebay, B.A.)—A thin conical vessel is filled with fluid and placed on a horizontal plane. Find where a small orifice must be made in the surface so that the issuing jet may fall at the foot of the cone. If m be the distance of this point from the vertex, show the average range *on the cone* will be equal to m if the semi-vertical angle of the cone be $\tan^{-1}\sqrt{2}$.

Solution by the Proposer.

Let l be the length of the slant side of the cone, $2a$ the vertical angle, z any distance from the vertex, and v the velocity of the issuing fluid. Then $v^2 = 2gz\cos a$, and the equation to the path of the jet is

$$y = x\tan a - \frac{gx^2}{2v^2\cos^2 a} = x\tan a - \frac{x^2}{4z\cos^3 a},$$

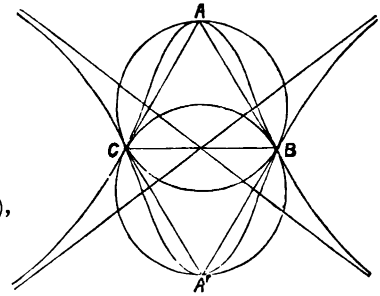
or

$$x^2 - 4zx\sin a\cos^2 a = -4zy\cos^3 a \quad\dots\dots\dots\dots(1).$$

Let $z = m$, and take $x = (l-m)\sin a$, $y = -(l-m)\cos a$; then, substituting in (1), we find $m = \dfrac{l\tan^2 a}{4+\tan^2 a}$. Let R be the range on the surface; then $x = R\sin a$, $y = -R\cos a$. Substituting again in (1), we find $R = 4z\cot^2 a$. Therefore $\int_0^m R\,dz = 2m^2\cot^2 a$, and the average range $= 2m^2\cot^2 a/m = 2m\cot^2 a$. Hence, if $2m\cot^2 a = m$, we have $a = \tan^{-1}\sqrt{2}$.

9595. (Asparagus.)—Given the base BC of a triangle in position and magnitude, find the locus of the vertex A, so that the distance between the circumcentre and orthocentre of the triangle ABC may be equal to the sum or difference of the sides AB, AC.

Solution by Professor Wolstenholme, M.A., Sc.D.



If A be the distance between the orthocentre and circumcentre, R the radius of the circumcentre,

$A^2 = R^2(1-8\cos A\cos B\cos C)$

$= R^2[1-4\cos B\cos C$

$\quad + (1-2\cos A)\times 4\cos B\cos C]$

$= R^2[2-2\cos(B-C)]$

$+ R^2(2\cos A - 1)(1-4\cos B\cos C)$,

and $\quad (b-c)^2$

$\quad = 4R^2(\sin B - \sin C)^2$

$\quad = 16R^2\sin^2\tfrac{1}{2}(B-C)\sin^2\tfrac{1}{2}A$

$\quad = 4R^2\sin^2\tfrac{1}{2}(B-C) + 4R^2\sin^2\tfrac{1}{2}(B-C)(1-2\cos A)$;

hence $\Delta^2 = (b-c)^2$, if $2\cos A = 1$, or if

$$1-4\cos B\cos C + 4\sin^2\tfrac{1}{2}(B-C) = 0.$$

Again, $\Delta^2 = R^2\left[1+4\cos B\cos C - (1+2\cos A)\times 4\cos B\cos C\right]$

$\qquad = R^2\left[2+2\cos(B-C)\right] - R^2(1+2\cos A)(1+4\cos B\cos C)$,

and $\quad (b+c)^2 = 4R^2(\sin B+\sin C)^2 = 16R^2\cos^2\tfrac{1}{2}(B-C)\cos^2\tfrac{1}{2}A$

$\qquad = 4R^2\cos^2\tfrac{1}{2}(B-C) + 4R^2\cos^2\tfrac{1}{2}(B-C)(1+2\cos A)$;

hence $\Delta^2 = (b+c)^2$, if $2\cos A + 1 = 0$, or if

$$1+4\cos B\cos C + 4\cos^2\tfrac{1}{2}(B-C) = 0.$$

Thus $\Delta = b \pm c$, if $A = 60°$ or $120°$, or if

$$\pm 3 + 6\cos B\cos C + 2\sin B\sin C = 0.$$

Thus the locus of A under the given conditions is two circles through B, C circumscribing the two equilateral triangles on BC, and a quartic whose equation (taking $2a$ for the length of BC, origin at the middle point of BC, and axis of x along BC) is

$$4(3a^2-3x^2+y^2)^2 = 9\left[(a^2+x^2+y^2)^2 - 4a^2x^2\right].$$

(If $BA = r_1$, $CA = r_2$, $\cos B = \dfrac{a-x}{r_1}$, $\sin B = \dfrac{y}{r_1}$, $\cos C = \dfrac{a+x}{r_2}$, $\sin C = \dfrac{y}{r_2}$, so that $6\cos B\cos C + 2\sin B\sin C = \dfrac{6(a^2-x^2)+2y^2}{r_1r_2}$, whence the equation.)

This quartic touches both the circles at the angular points of the two equilateral triangles on BC, its radius of curvature at B or C being three times the radius of either circle and in the opposite sense; at the other corners of the equilateral triangles its radius of curvature is one-fifth of the radius of the circle and in the same sense. There are two real asymptotes $5y^2 = 3x^2$, each meeting the curve in two finite points,

where $y^2 = \frac{2}{13}a^2$. The curve may readily be drawn from these data. If we refer to one of the equilateral triangles for areal or trilinear coordinates, the two circles are $-S_1 \equiv yz+zx+xy = 0$, $S_2 \equiv x^2-yz$, and the quartic is $\quad U \equiv (yz+zx+xy)(x^2-yz)+4yz(x+y)(x+z) = 0.$

Hence, if from any point P on the quartic we draw PO, PO′ tangents to the two circles, and PQ, PQ′, Pq, Pq' perpendiculars on the sides (excluding BC) of the equilateral triangles,

$$\text{PO}\,.\,\text{PO}' = \tfrac{4}{3}\,(\text{PQ}\,.\,\text{PQ}'\,.\,\text{P}q\,.\,\text{P}q')^{\frac{1}{2}}.$$

The points B, C are single foci of this quartic, not double as might seem at first sight to be the case, since if

$$x-a+iy = 0,\quad r_1 = 0,\quad \text{and}\quad (3a^2-3x^2+y^2)^2 = 0,$$

but the point $(a,\,0)$ is a node on the quartic, and the only point of contact of this focal tangent is when $x = \frac{1}{2}a$.

[If K be the orthocentre, I the circumcentre, O the incentre, O_1, O_2, O_3 the three excentres, then when $A = 60°$, the angles of the triangle KIO are

$$180°-\tfrac{1}{2}(B\sim C),\quad \tfrac{1}{4}(B\sim C),\quad \tfrac{1}{4}(B\sim C);$$

and when $A = 120°$, the angles of the triangle KIO_2 are

$$90°+\tfrac{1}{2}(B-C),\quad 45°-\tfrac{1}{4}(B-C),\quad 45°-\tfrac{1}{4}(B-C),$$

and those of the triangle KIO_3 are

$$90°-\tfrac{1}{2}(B-C),\quad 45°+\tfrac{1}{4}(B-C),\quad 45°+\tfrac{1}{4}(B-C).$$

The equilateral triangle aBC being the triangle of reference, the two circles $-S_1 \equiv yz+zx+xy = 0$, $S_2 = x^2-yz = 0$, and the quartic

$$U \equiv (yz+zx+xy)(x^2-yz)+4yz(x+y)(x+z) = 0$$

are together the locus of the vertex of a triangle of which BC is one side, and such that the distance between the circumcentre and orthocentre is equal to the difference or the sum of the two sides other than BC. It may perhaps deserve mention that any quartic whose equation is

$$(yz+zx+xy)(x^2-yz)+kyz(x+y)(x+z) = 0$$

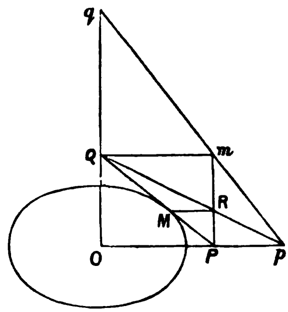
will touch the circles at aBCa', the radius of curvature of the quartic at B, C on either branch being $(1-k)$ times the radius of the circle, and the curvature of the quartic at a, a' being $(1+k)$ times the curvature of the circle. If $k = 1$, the quartic breaks up $x^2(2yz+zx+xy) = 0$. It would be interesting to find what is the locus of the inflexions of this quartic for different values of k.]

9620. (Professor De Longchamps.)—On considère deux axes Ox, Oy et deux points $m\,(x,\,y)$, M (X, Y) qui se correspondent de telle sorte que l'on ait $\qquad xX = a^2,\quad yY = b^2$ ....(1),

a, b désignant deux constantes données. Si m décrit une courbe U, le point correspondant M décrit une autre courbe V; les tangentes à ces courbes, aux points m, M, coupent les axes respectivement aux points p, q; P, Q. Démontrer que l'on a $mp : mq = \text{MP} : \text{MQ}$. Déduire, de là, le tracé par points et par tangentes des courbes qui se correspondent dans la *transformation réciproque cartésienne*, que définissent les formules

(1). Appliquer la propriété en question aux courbes représentées par l'équation $x^2y^2 = Ax^2 + By^2$.

Solution by Professor GENESE, M.A.



Taking logarithms and differentiating, we have

$$\frac{dx}{x} + \frac{dX}{X} = 0, \quad \frac{dy}{y} + \frac{dY}{Y} = 0;$$

therefore $\quad \dfrac{y\,dx}{x\,dy} = \dfrac{Y\,dX}{X\,dY}.$

Now $\dfrac{mp}{mq} = \dfrac{y \operatorname{cosec} qpO}{x \sec qpO}$

$$= \frac{y}{x} \cot qpO = -\frac{y}{x} \cdot \frac{dx}{dy}, \text{ \&c.}$$

Again, let y meet pQ at R, then

$$pR : RQ = pm : mq = PM : MQ;$$

therefore RM is parallel to pP. This gives an obvious construction for PQ if pq be known.

The curve $\dfrac{A}{y^2} + \dfrac{B}{x^2} = 1$ is transformed into $\dfrac{A}{b^4} Y^2 + \dfrac{B}{a^4} X^2 = 1$,

or, choosing $b^2 = A$, $a^2 = B$, $\quad \dfrac{Y^2}{b^2} + \dfrac{X^2}{a^2} = 1.$

In this case perpendiculars to the axes through P, Q meet at m, and points and tangents of the conic immediately yield points and tangents to the curve.

9585. (Professor NILKANTHA SARKAR, M.A.)—BC is a side of a square; on the perpendicular bisector of BC two points P, Q are taken equidistant from the centre of the square; BP, CQ are joined, and cut in A; prove that, in the triangle ABC, $\tan A (\tan B - \tan C)^2 + 8 = 0$.

Solution by T. YOUNG, M.A.; W. J. GREENSTREET, B.A.; *and others.*

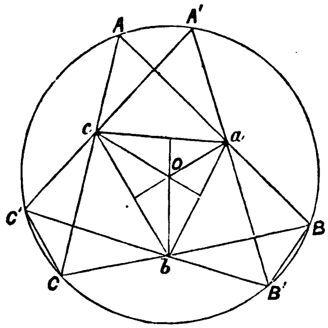
Let $BC = 2a$, and $PQ = 2ma$; then

$\tan B = 1 + m, \quad \tan C = 1 - m, \quad$ and $\tan A = -\tan(B + C) = -2/m^2$.

Thus $\qquad \tan A (\tan B - \tan C)^2 = -2/m^2 \times 4m^2 = -8.$

9630. (The EDITOR.)—If a triangle ABC turns around its circumcentre O into the position A′B′C′, and if AB, A′B′ meet in a; BC, B′C′ in b, and CA, C′A′ in c, prove that the triangle abc will have O for its *orthocentre.*

Solution by Professors Beyens, De Wachter, *and others.*



On a arc $AA' = BB' = CC'$, et angle $BaB' = BbB'$, et le quadrilatère $abBB'$ inscriptible donnera $abB = aB'B = \frac{1}{2}$ (arc AB − arc AA'); aussi $cbC = 180^\circ - CC'c$

$= 180^\circ - \frac{1}{2}$ (arc CB + arc AB − arc AA'),

et $abc = 180^\circ - abB - cbC$

$= \frac{1}{2}$ arc CB = angle BAC.

Mais l'angle

$AOA' = AcA' = AaA'$,

donc le point O est sur la circonférence $AA'ca$, et par suite

$aoc = 180^\circ - A = 180^\circ - abc$;

de la même manière on a $boc = 180^\circ - bac$, $aob = 180^\circ - acb$, c'est à dire que O est l'orthocentre de abc.

[Referring to the solution of Quest. 9508 (Vol. L., p. 40), and using the same notation, we have

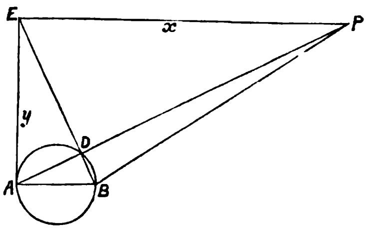
$$Oa = (\alpha+\beta)\frac{i^n}{1+i^n},\quad Ob = (\beta+\gamma)\frac{i^n}{1+i^n},\quad Oc = (\gamma+\alpha)\frac{i^n}{1+i^n};$$

hence, $$Oa - Oc = ca = (\beta-\gamma)\frac{i^n}{1+i^n};$$

therefore Ob is perpendicular to ca, and, similarly, Oa, Oc will cut cb, ba at right angles.]

9634. (Professor De Wachter.)—Given a circle and its diameter AB. D being any point in the circumference, BD is drawn to meet in E the tangent at A. The perpendicular drawn from E to AE cuts AD in P. Required the locus of P.

Solution by Professor Schoute.

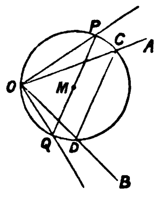


The quadrilateral ABPE is a rectangular trapezoid, the diagonals of which are perpendicular to one another. Therefore $AE^2 = AB \cdot EP$, or with reference to AB and AE as axes of coordinates, and when p is the radius of circle AB, $y^2 = 2px$, etc.

9616. (Professor Mannheim.)—On donne un angle. Par le sommet de cet angle on fait passer un circonférence quelconque, et l'on joint par

une droite les points où elle rencontre les côtés de l'angle. Le diamètre parallèle à cette droite coupe la circonférence en deux points, dont on demande le lieu lorsqu'on fait varier cette courbe.

Solution by J. C. ST. CLAIR, M.A.; Professor SCHOUTE; *and others.*



When AOB is the given angle, COD the given circle, and PMQ its diameter parallel to CD, it is evident that the angles POA and BOQ are equal. These angles are constant too, as the angles AOB and POQ = 90° are constant. The locus consists of the two lines OP and OQ forming a right angle, the bissectrices of which coincide with those of AOB.

9611. (Professor HUDSON, M.A.)—A particle A moves in a straight line, and a second particle B always moves towards A and keeps at a constant distance from it. Find (1) the path of B, and show (2) that its velocity is a mean proportional between the velocity of its projection on the path of A and the velocity of A.

Solution by Professors DE WACHTER, NILKANTHA SARKAR, *and others.*

OX being taken along A's path, and OY perpendicular to it, let x, y denote the coordinates of B, and a the invariable distance BA. The tangent to B's trajectory being $= a$, we have $dy/dx = \pm\, y/(a^2-y^2)^{\frac{1}{2}}$. Considering the branch to the right of the cusp in the OY-axis ($y = a$, $x = 0$), we have to take the minus sign. By integration we get

$$x = a \log\left\{\frac{a+(a^2-y^2)^{\frac{1}{2}}}{y}\right\} - (a^2-y^2)^{\frac{1}{2}},$$

which is the equation to a *tractrix*.

The velocity of A is the resultant of two rectangular velocities, the one perpendicular to BA, the other along BA, which is the velocity of B itself. Thus, if AbA' be a rectangular triangle whose hypotenuse AA' represents A's velocity, bA' being the velocity of B, and $b'A'$ its projection on AA', we have $(bA')^2 = b'A'.AA'$.

9471. (E. W. SYMONS, M.A.)—To expand $\cos x$ and $\sin x$ in series of ascending powers of x, by a process more concise than the expansions given in the ordinary text-books.

Solution by the PROPOSER.

Let $F(x) \equiv \cos x + i \sin x$ (where $i \equiv \sqrt{-1}$); then we have at once, by DE MOIVRE's theorem, $F(x+y) = F(x) \times F(y)$; therefore we may

assume $F(x) = e^{f(x)}$, and we have to determine $f(x)$. Now, by DE MOIVRE,

$$\{F(x)\}^n = F(nx), \text{ or } e^{nf(x)} = e^{f(nx)}; \text{ therefore } f(nx) = nf(x).$$

This at once leads to $f(x) \equiv kx$, therefore k having to be determined, we have proved that $\cos x + L \sin x = e^{kx}$ ....(1).

Now since $\cos x = \cos(-x)$ and $\cos 0 = 1$, we may assume

$$\cos x = 1 - \frac{a_2 x^2}{2!} + \frac{a_4 x^4}{4!} \ldots\ldots,$$

and since $\sin x = -\sin(-x)$ and $\dfrac{\sin x}{x} = 1$, when $x = 0$, we may assume

$$\sin x = \frac{x}{1!} - \frac{a_3 x^3}{3!} + \frac{a_5 x^5}{5!} \ldots\ldots.$$

(1) may be written

$$1 - \frac{a_2 x^2}{2!} + \frac{a_4 x^4}{4!} - \ldots\ldots + i\left(\frac{x}{1!} - \frac{a_3 x^3}{3!} + \frac{a_5 x^5}{5!} \ldots\right)$$

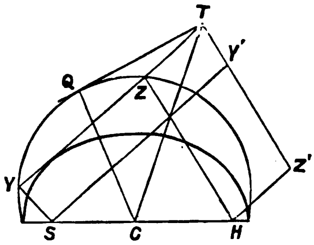
$$\equiv e^{kx} = 1 + \frac{kx}{1!} + \frac{k^2 x^2}{2!} + \frac{k^3 x^3}{3!} + \frac{k^4 x^4}{4!} + \frac{k^5 x^5}{5!} \ldots\ldots;$$

therefore, equating coefficients, $k = i$, $a_2 = 1$, $a_3 = 1$, $a_4 = 1$, $a_5 = 1$, &c.;

therefore $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \ldots$, and $\sin x = \dfrac{x}{1!} - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots$.

9653. (Rev. T. ROACH, M.A.)—Two ellipses have their foci coincident. Find the locus of the intersection of those tangents which cut at right angles.

Solution by A. E. THOMAS; Rev. J. J. MILNE, M.A.; *and others.*



T is the point of intersection of tangents at right angles, and TQ is the tangent from T to the inner auxiliary circle. Then, with the usual construction,

$$CT^2 = CQ^2 + QT^2 = CQ^2 + TZ \cdot TY$$
$$= CQ^2 + HZ' \cdot SY' = \alpha^2 + \beta^2;$$

hence the locus of T is a circle.

9441. (W. J. GREENSTREET, B.A.)—Find the n^{th} pedal of

$$r^m = a^m \cos m\theta,$$

and the radius of curvature of the $(m-1)^{\text{th}}$ pedal.

Solution by Professor MADHAVARAO, M.A.

The equation $r^m = a^m \cos m\theta$ after transformation becomes $r^{m+1} = a^m p$; and the equations to the successive pedals are

$$r^{2m+1} = a^m p^{m+1}, \quad r^{3m+1} = a^m p^{2m+1}, \quad r^{4m+1} = a^m p^{3m+1},$$

and so on, and therefore the equation to the n^{th} pedal is

$$r^{(n+1)m+1} = a^m p^{nm+1}.$$

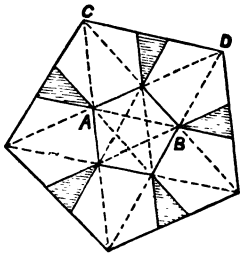
The $(m-1)^{\text{th}}$ pedal is $r^{m^2+1} = a^m p^{m^2-m+1}$; and its radius of curvature is

$$\rho = r\frac{dr}{dp} = \frac{m^2-m+1}{m^2+1}\, a^{m/(m^2-m+1)}\, r^{(m-1)^2/(m^2-m+1)}.$$

9454. (Professor HANUMANTA RAU, M.A.)—Show (1) how to cut out six equal regular pentagons from a given regular pentagon; and (2) find the area of the portion left out.

Solution by Professor SCHOUTE.

The solution is given in the adjoined figure, obtained by drawing the diagonals in the given pentagon, and prolonging the diagonals AB of the little included pentagon. We find



$$\frac{\text{AB}}{\text{CD}} = \tfrac{1}{2}(3-\sqrt{5}),$$

and

$$\frac{\text{surface left out}}{\text{surface given pentagon}} = 9\sqrt{5}-20 = 0,\ 12461\ \ldots$$

9387. (Professor SWAMINATHA AIYAR, B.A.)—$\text{F}(x^2) = f(x) \,.\, f(-x)$; and $\text{F}(2-x^2)$ and $f(x)$ have ax^3+bx^2+cx+d for their G. C. M.; show that

$$(b^2-2a^2-ac)^2 = a^2(b+d)^2.$$

Solution by the PROPOSER.

$\text{F}(2-x^2) = f[(2-x^2)^{\frac{1}{2}}] \,.\, f[-(2-x^2)^{\frac{1}{2}}]$, and is therefore the rationalised form of $f[(2-x^2)^{\frac{1}{2}}]$. Therefore the roots $f[(2-x^2)^{\frac{1}{2}}] = 0$ are identical with those of $\text{F}(2-x^2) = 0$. Again, if $f(x) = 0$ and $f[(2-x^2)^{\frac{1}{2}}] = 0$ have a common root a (different from $1^{\frac{1}{2}}$) they must have another common root $(2-a^2)^{\frac{1}{2}}$. As they have just three roots in common, these must be $1^{\frac{1}{2}}$, a, $(2-a^2)^{\frac{1}{2}}$, and the product of factors corresponding to these roots must be of the form

$$(x^2+2px+2p^2-1)(x-1^{\frac{1}{2}}) = x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}.$$

If we take the positive value of $1^{\frac{1}{4}}$, we have

$$2p-1 = \frac{b}{a}, \quad 2p^2-2p-1 = \frac{c}{a}, \quad \text{and} \quad 1-2p^2 = \frac{d}{a},$$

and consequently $\quad b^2-2a^2-ac = -a(b+d)$;

if we take the negative value, we have $b^2-2a^2-ac = a(b+d)$; both which relations are included in $(b^2-2a^2-ac)^2 = a^2(b+d)^2$.

The first n natural numbers are written down in a random order. Find the chance that no number is immediately followed or preceded by its next higher number.

Let ${}_nP_4$ represent the number of permutations of n numbers (consisting of the first four natural numbers and $n-4$ others, no one of which is consecutive with any of the other $n-1$) taken altogether, in which no number is immediately followed or preceded by the next higher number. To find ${}_nP_5$. In ${}_nP_4$ change (m) one of the $n-4$ non-consecutive numbers into 5, and reject all those which contain the element 45 or 54. Those which contain the element 45 are ${}_{n-1}P_4 + {}_{n-2}P_3 + {}_{n-3}P_2 + {}_{n-4}P_1$, and there are as many which contain the element 54. Therefore

$${}_nP_5 = {}_nP_4 - 2\left({}_{n-1}P_4 + {}_{n-2}P_3 + {}_{n-3}P_2 + {}_{n-4}P_1\right),$$

and generally ${}_nP_{r+1} = {}_nP - 2\left({}_{n-1}P_r + {}_{n-2}P_{r-1} + {}_{n-3}P_{r-2} \ldots + {}_{n-r}P_1\right)$,

and $\quad {}_{n-1}P_r = {}_{n-1}P_{r-1} - 2\left({}_{n-2}P_{r-1} + {}_{n-3}P_{r-2} \ldots + {}_{n-r}P_1\right)$.

Hence we have $\quad {}_nP_{r+1} = {}_nP_r - {}_{n-1}P_r - {}_{n-1}P_{r-1}$.

But ${}_nP_0 = n!$ and ${}_nP_1 = n!$. Therefore

$${}_nP_2 = n! - 2(n-1)!, \quad {}_nP_3 = n! - 4(n-1)! + 2(n-2)!,$$

$${}_nP_4 = n! - 6(n-1)! + 8(n-2)! - 2(n-3)!,$$

and ${}_nP_n = F_0\, n! - F_1 (n-1)! + F_2 (n-2)! \ldots (-1)^r F_r (n-r)! + \&c. \&c.$,

where $F_r = 2^r . {}_{n-1}C_r - 2^{r-1} . {}_{r-1}C_1 . {}_{n-1}C_{r-1} + 2^{r-2} . {}_{r-1}C_2 . {}_{n-1}C_{r-2} - \&c. \&c.$,

$$(-1)^p . 2^{r-p} . {}_{r-1}C_p . {}_{n-1}C_{r-p}, \ \&c.,$$

and the required chance is $\frac{{}_nP_n}{n!}$. It can easily be proved that when n is infinite the chance is e^{-2}.

9554. (R. Knowles, B.A.)—Two rectangular hyperbolas touch at a point P and meet again in one other point Q; prove that, (1) Q is on the normal at P, (2) the locus of the centres is a circle whose diameter is equal to the radius of curvature at P.

Solution by Professors Abinash Basu, Matz, *and others.*

The hyperbolas being referred to the tangent and normal as axes, let their equations be

$$x^2 + 2hxy - y^2 + 2gx = 0, \quad x^2 + 2h'xy - y^2 + 2fx = 0 \ldots\ldots\ldots(1, 2).$$

Now, by the Question, they have the same curvature at P, therefore $g = f$. Subtracting (2) from (1), we get $(h-h')\,xy = 0$, which shows that Q is on the normal at P.

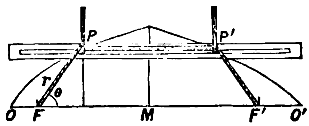
In (2), taking h' to be variable and representing (2) by ϕ, we get

$$\frac{d\phi}{dx} \equiv 2x+2h'y+2g=0, \qquad \frac{d\phi}{dy} \equiv -2y+2h'y+x=0.$$

Eliminating h', we get the locus circle.

9487. (J. O'Byrne Croke, M.A.)—Two pencils are made to move towards each other in a smooth straight slot in a bar by a string which passes tensely and symmetrically round them with its ends fastened to two fixed pins, as the bar moves outwards so as to be always parallel to the line joining the pins; determine the paths of the tracing points.

Solution by the Proposer.



Let O, O′ represent the initial position of the bar, F, F′ the fixed pins to which the extremities of the string are tied, P, P′ points in the traces of the pencils. Then, if F, F′ $= 2b$, and the whole length of the string be $4a+2b$, we have, for polar coordinates,

$$r+(b-r\cos\theta)=2a+b; \text{ therefore } r(1-\cos\theta)=2a, \quad r=2a/(1-\cos\theta).$$

Hence, the traces are portions of parabolas having F, F′ as foci, and O, O′ as vertices.

[Mr. Carr remarks that "it is obvious that the curve is a parabola: for the slot being, say, of the same length MM′ as the string, PF = PM, or distance from focus = distance from directrix. The apparatus is simply the ordinary parabolic compass (described in my *Synopsis* at 1249) duplicated. It is, however, useless as described, for there is no contrivance for keeping the bar parallel to FF′, nor for keeping PF = P′F′."]

8443. (Professor Mahendra Nath Ray, M.A.)—If p be the pole of the small circle circumscribing an equilateral spherical triangle ABC, and L any other point on the sphere; and if $\cos AL = x$, $\cos BL = y$, $\cos CL = z$, $LP = \lambda$, $AP = R$, $\angle APL = \theta$, show that

$$27\sin^3\lambda\sin^3 R = 4\sec 3\theta\,(2x-y-z)(2y-z-x)(2z-x-y).$$

Solution by G. G. Storr, M.A., *and* Professor Ignacio Beyens.

We have $x = \cos R\cos\lambda+\sin R\sin\lambda\cos\theta$; and y and z can be obtained by writing for θ, $120+\theta$ and $120-\theta$, respectively; therefore

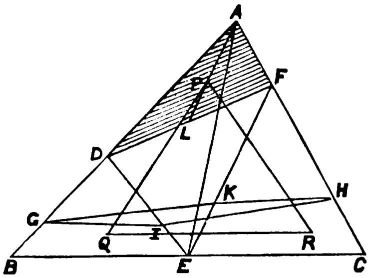
$$2x-y-z=2\sin R\sin\lambda\cos\theta, \quad 2y-z-x=-\tfrac{3}{2}\sin R\sin\lambda(\cos\theta-\sqrt{3}\sin\theta),$$

$$2z-x-y=-\tfrac{3}{2}\sin R\sin\lambda(\cos\theta+\sqrt{3}\sin\theta);$$

hence, multiplying these equations together, we get the result.

3178. (Professor ASHER B. EVANS, M.A.)—The point D is taken at random in the side AB of a triangle ABC, E in BC, and F in CA: the point P is taken at random in the triangle ADF, Q in BDE, and R in CEF. The average area of the triangle PQR is required.

Solution by D. BIDDLE.



Without needing integration in the ordinary sense of the term, this question affords a good illustration of the principle which governs the reduction of a multiple integral.

In the first place, let D, E, F be regarded as fixed, also Q, R; let P be anywhere on the line DF. It is evident that the mean value of PQR will then be given by LQR, L being the mid-point of DF. Similarly, for any line in ADF, parallel to DF, AL will be the locus of the vertex of the mean triangle, and it is easy to see that the centroid of ADF, which is two-thirds the distance from A to L, will be the mean position of P. Likewise the centroids of BDE, CEF will be the mean positions of Q, R, respectively.

In the next place, in order to see the effect of moving the random points on the perimeter, let D, F be regarded as fixed, whilst E is moved along BC. The locus of the centroid of BDE will be GI, and that of CEF will be HK, both being lines parallel to BC, and two-thirds the distance from the respective apices. When one centroid is at G, the other will be at K, and they will travel along their respective lines at the same rate, so that the line joining them is always parallel to GK, and the perpendicular from P upon GK is also perpendicular to HI and to every line of junction between the two. PGK is the smallest triangle, and PHI the largest; and the mean triangle is mid-way between the two, that is, when R, Q bisect HK, GI, respectively, and E is consequently midway along BC. The mean positions for D, F will be midway on their respective lines also. And if we now join them, and place P, Q, R at the respective centroids of the triangles belonging to them, we shall, on joining PQR, have the mean triangle required. The sides of this triangle are parallel to those of ABC, but half the length; therefore its area $= \frac{1}{4}$ABC, which is the area of the mean DEF as well.

9563. (ARTEMAS MARTIN, LL.D.)—Find six whole positive numbers the sum of whose fifth powers is a fifth power.

Solution by the PROPOSER.

I am not aware that any general method has yet been discovered of solving, when $n > 3$, the equation $x_1^n + x_2^n + x_3^n + \ldots + x_m^n = y^n$; but, by the

following tentative process, I have succeeded in finding values of x_1, x_2, x_3, &c., and y, for $n = 2, 3, 4, 5$; and for several values of m greater than 6, when $n = 5$.

Put $1^n + 2^n + 3^n + 4^n + \ldots + x^n = S_{x,n}$; from $S_{x,n}$ subtract b^n $(b > x)$, and put the remainder $= r$; then we have $S_{x,n} - b^n = r$, or $S_{x,n} - r = b^n$.

Now, if r can be separated by trial into nth-power numbers, all different and none greater than x^n we have

$$1^n + 2^n + 3^n + 4^n + \ldots + x^n - (\text{these } n\text{th-power numbers}) = b^n.$$

When $n = 5$, we have $S_{x,5} = \frac{1}{12}x^2(x+1)^2(2x^2 + 2x - 1)$.

Examples.—1. Take $x = 11$, then $S_{x,5} = 381876$; take $b = 12$, then

$$r = 133044 = 10^5 + 8^5 + 3^5 + 2^5 + 1^5;$$

therefore $$1^5 + 2^5 + 3^5 + \ldots + 11^5 - (1^5 + 2^5 + 3^5 + 8^5 + 10^5)$$
$$= 4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5.$$

2. Take $x = 29$, then $S_{x,5} = 109687425$; take $b = 30$, then

$$r = 85387425 = 28^5 + 27^5 + 26^5 + \ldots + 20^5 + 18^5 + 17^5 + 15^5$$
$$+ 14^5 + 13^5 + 12^5 + 9^5 + 8^5 + 7^5 + 6^5 + 4^5 + 3^5 + 2^5 + 1^5;$$

therefore $$S_{x,5} - r = 5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5.$$

Otherwise:—Assume $p^n - q^n = d$, then $q^n + d = p^n$, and if d can be separated by trial into nth-power numbers, all different, and q^n not among them, we have $q^n +$ (these nth-power numbers) $= p^n$.

Example.—Take $p = 30$, $q = 29$;

then $$d = 3788851 = 19^5 + 16^5 + 11^5 + 10^5 + 5^5;$$

therefore $$5^5 + 10^5 + 11^5 + 16^5 + 19^5 + 29^5 = 30^5.$$

This method is preferable when only a few numbers are sought: but when a large *number* of numbers is required it will be better to use the first method.

I have found many sets of numbers for other values of m greater than 6, but only these two for $m = 6$; but there are doubtless a great number of such sets.

I do not know that any other person besides myself has succeeded in finding fifth-power numbers whose sum is a fifth power. If any such numbers have been previously found I will be glad to be apprised of the fact—and the numbers.

9497. (Professor ABINASH CHANDRA BASU, M.A.)—A point moves such that the triangle formed by the two tangents from it to a conic and the chord of contact is constant. Prove that its locus is a similar and similarly situated conic.

Solution by R. KNOWLES, B.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

In the case of central conics the area of the triangle in question is readily found to be $(a^2y^2 + b^2x^2 - a^2b^2)^{\frac{3}{2}} / (a^2y^2 + b^2x^2)$, and if this is constant $a^2y^2 + b^2x^2 =$ const. In the case of the parabola the area is $(y^2 - 4ax)^{\frac{3}{2}} / a$, and $y^2 - 4ax =$ const.

9627. (Professor De Wachter.)—A_1, A_2 and B_1, B_2 are two couples of points respectively taken in OX and OY, A_1B_1 and A_2B_2 meeting in C, A_1B_2 and A_2B_1 in D. If, OX being fixed, OY revolves about O in the plane; find (1) the loci of C and D; (2) the envelope of CD.

Solution by Rev. J. J. Milne, M.A.; J. C. St. Clair, M.A.; *and others.*

1. C and D are centres of projection when we take (A_1, B_1), (A_2, B_2), and (A_1, B_2), (A_2, B_1) respectively as pairs of corresponding points. Draw DP, CQ parallel to OY meeting OX in P and Q. Then the loci of C and D are obviously circles, centres Q and P respectively.

2. Let CD meet OX in E; then

$$EP : PD = EQ : QC = PQ : QC - PD.$$

Hence EP is constant, and therefore CD always passes through the fixed point E.

9570. (J. Brill, M.A.)—ABC is a portion of a thin rigid spherical shell bounded by arcs of great circles. It lies in equilibrium on a horizontal plane, the curved surface being in contact with the plane. Prove that, if O be the centre of the surface, and

$$Q^2 = a^2 + b^2 + c^2 - 2bc\cos A - 2ca\cos B - 2ab\cos C,$$

the cosines of the angles that OA, OB, OC make with the vertical are respectively $\dfrac{a}{Q} \cdot \dfrac{\sin b \sin c \sin A}{\sin a}$, $\dfrac{b}{Q} \cdot \dfrac{\sin c \sin a \sin B}{\sin b}$, $\dfrac{c}{Q} \cdot \dfrac{\sin a \sin b \sin C}{\sin c}$.

Solution by W. S. Foster, M.A.; Prof. Ignacio Beyens; *and others.*

Let G be the centre of gravity of ABC, then OG will be vertical; hence, if O be the origin, OA the axis of x, AOB the plane of xy, and x, y, z the coordinates of G, we have

$$x = \tfrac{1}{2}r\,\frac{a\sin B\sin c}{A+B+C-180^\circ},\qquad y = \tfrac{1}{2}r\,\frac{b\sin A - a\sin B\cos c}{A+B+C-180^\circ},$$

$$z = \tfrac{1}{2}r\,\frac{c - b\cos A - a\cos B}{A+B+C-180^\circ},$$

(see Walton's *Mechanical Problems*, p. 30, ed. 1842). Therefore

$$\cos GOA = \frac{x}{(x^2+y^2+z^2)^{\frac{1}{2}}} = \frac{a\sin B\sin c}{Q} \text{ or } = \frac{a}{Q} \cdot \frac{\sin b\sin c\sin A}{\sin a},$$

and similarly for the other angles.

9602. (E. Rutter.)—A pipe of water, flowing uniformly at the rate of a gallon a minute, falls into a six-gallon cask half full of wine; find (1) how long the water must flow into the cask so that the quantity of pure wine may be reduced to one gallon, and (2) in what time the wine will be wholly removed from the cask if the pipe be kept running.

Solution by J. YOUNG, M.A.; SARAH MARKS, B.Sc.; *and others.*

1. The replacing of a drop d of the mixture by a drop of water reduces the quantity of wine from v to $v\,(1-\frac{1}{6}d)$. If there are n drops in a gallon, $nd = 1$, and the quantity of wine remaining after the first minute of overflow is the limiting value, when n is infinite, of the expression $3 \times (1-\frac{1}{6}d)^n$ or $3e^{-\frac{1}{6}}$.

If t denote the time at which one gallon remains, $3\,e^{-\frac{1}{6}t} = 1$; whence

$$t = 6 \log 3 / \log e = 6 \text{ minutes } 35\tfrac{1}{2} \text{ seconds.}$$

2. The time required for the total removal of the wine is theoretically infinite; but at the end of an hour and twenty minutes the wine is less than one part in one million—a safe temperance drink.

9590. (A. RUSSELL, B.A.)—If a circle can be inscribed in a pentagon ABCDE, prove that

$$(a-c)\cot\tfrac{1}{2}\mathrm{E} + (b-d)\cot\tfrac{1}{2}\mathrm{A} + (c-e)\cot\tfrac{1}{2}\mathrm{B} + (d-a)\cot\tfrac{1}{2}\mathrm{C} + (e-b)\cot\tfrac{1}{2}\mathrm{D}$$
$$= (a-e)\cot\tfrac{1}{2}\mathrm{A} + (b-a)\cot\tfrac{1}{2}\mathrm{B} + (c-b)\cot\tfrac{1}{2}\mathrm{C} + (d-c)\cot\tfrac{1}{2}\mathrm{D} + (e-d)\cot\tfrac{1}{2}\mathrm{E} = 0.$$

Solution by J. YOUNG, M.A.; Rev. J. L. KITCHIN, M.A.; *and others.*

Changing a into $\mathrm{R}\,(\cot\frac{1}{2}\mathrm{A} + \cot\frac{1}{2}\mathrm{B})$, where R is the radius of the circle, with similar values for all the other sides, the given expressions vanish identically.

9514. (J. O'BYRNE CROKE, M.A.)—If from all points of the section of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 - 1 = 0$ by the plane (yz) straight lines be drawn of length a to meet the axis of x; prove that they lie upon the surface $x^2 = \left\{a^2 - (y^2+z^2) \Big/ \left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)\right\} \left\{1 \pm \left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)^{\frac{1}{2}}\right\}^2.$

Solution by Professor ABINASH CHANDRA BASU, M.A.

Let $(0,\ b\cos\alpha,\ c\sin\alpha)$ and $(m,\ 0,\ 0)$ be the two points whose join is of length a; then the equations to the join are

$$\frac{x-m}{m} = \frac{y}{-b\cos\alpha} = \frac{z}{-c\sin\alpha} = \pm\left(\frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^{\frac{1}{2}} \quad \text{............(1),}$$

also $$m^2 + b^2\cos^2\alpha + c^2\sin^2\alpha = a^2 \quad \text{..........................(2).}$$

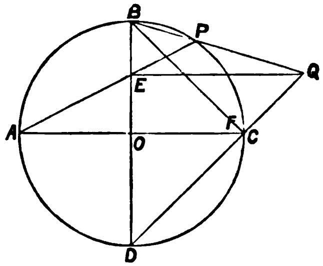
Now, from equations (1) we get

$$m = \frac{x}{1 \pm t}, \quad b\cos\alpha = \pm\frac{y}{t}, \quad \text{and} \quad c\sin\alpha = \pm\frac{z}{t},$$

where t stands for $\pm\left(\dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}\right)^{\frac{1}{2}}$. Substituting these in (2), we get the solution.

9591. (W. J. Greenstreet, M.A.)—AC, BD are diameters of a circle ABCD at right angles; P any point on the circumference; PA, BD intersect in E; EQ parallel to AC cuts PB in Q: prove that the locus of Q is a straight line.

Solution by Emily Perrin; Rosa Whapham; *and others.*



Let DC cut BP in Q, then the angles BEQ, BCQ are right angles, $\angle$ EQC = EBC = ACD = $\frac{1}{4}\pi$; therefore EQ, AC are parallel, *i.e.*, the locus of Q is the straight line DC.

[Hence, if BOD, AOC be any conjugate diameters of a conic, the property holds and may be deduced from the above by projection.]

9389. (Professor Hanumanta Rau.) — Prove (1) that $\sin 6°$ is a root of the equation $16x^4+8x^3-16x^2-8x+1=0$; and (2) express the remaining roots in terms of trigonometrical functions.

Solution by Professors Ignacio Beyens, Matz, *and others.*

Posons $x = z/2$; alors l'équation proposée est $z^4+z^3-4z^2-4z+1=0$. Changeons (z) en $(-z)$ et on aura $z^4-z^3-4z^2+4z+1=0$. Mais les racines de cette équation sont (*voyez* Serret. *Trig.*)

$$2\cos\frac{2\pi}{15},\ \ 2\cos\frac{4\pi}{15},\ \ 2\cos\frac{8\pi}{15},\ \ 2\cos\frac{14\pi}{15};$$

donc les racines de l'équation donnée sont

$$-\cos\frac{2\pi}{15},\ \ -\cos\frac{4\pi}{15},\ \ -\cos\frac{8\pi}{15}=\cos\frac{7\pi}{15}=\cos 84°=\sin 6°,\ \ -\cos\frac{14\pi}{15}.$$

7830. (R. Knowles, B.A., L.C.P.)—The line joining the centres of the in-circle and circum-circle of a triangle ABC meets BC in D, AB in F, and AC produced in E; if AD, FC intersect in H, prove that AB, BH, BC, BE form a harmonic pencil.

Solution by Professors Hanumanta Rau, Beyens, *and others.*

Taking the given triangle as the triangle of reference and using trilinear coordinates, the line passing through the in-centre (1, 1, 1) and circum-centre (cos A cos B cos C) is

$$(\cos B-\cos C)\,\alpha+(\cos C-\cos A)\,\beta+(\cos A-\cos B)\,\gamma=0\ \ldots\ldots\ldots(1).$$

The equation to AD is $(\cos C - \cos A)\,\beta + (\cos A - \cos B)\,\gamma = 0$, for it passes through the intersections of $\beta = 0$ and $\gamma = 0$, and (1) and $\alpha = 0$.

Similarly, the equations to CF, BE are

$$(\cos B - \cos C)\,\alpha + (\cos C - \cos A)\,\beta = 0 \quad\text{..................(2)},$$

$$(\cos B - \cos C)\,\alpha + (\cos A - \cos B)\,\gamma = 0 \quad\text{..................(3)}.$$

Equation to BH is obtained by eliminating β from (1) and (2), and is therefore $(\cos B - \cos C)\,\alpha - (\cos A - \cos B)\,\beta = 0$ ....(4).

AB, BH, BC, BE are represented by equations of the form $\gamma = 0$, $\alpha + \kappa\gamma = 0$, $\alpha = 0$, $\alpha - \kappa\gamma = 0$, and make up a harmonic pencil.

[The property just proved holds good in the case of *any* transversal cutting the sides in D, E, F. For, if $l\alpha + m\beta + n\gamma = 0$ be the transversal, BE is $l\alpha + n\gamma = 0$, and BH is $l\alpha - n\gamma = 0$.]

9372. (A. Russell, B.A.)—The particular integral of the equation $\dfrac{d^4u}{dx^4} - u = f(x)$ may be written $\frac{1}{2}\int_0^x \{\sinh(x-\xi) - \sin(x-\xi)\} f(\xi)\, d\xi$; write the particular integral of $\dfrac{d^4u}{dx^4} + u = f(x)$ in a similar form and hence solve completely the equation $\dfrac{d^8u}{dx^8} - u = f(x)$.

Solution by the Proposer.

The particular integral of $\dfrac{d^4u}{dx^4} + u = f(x)$ is

$$\frac{1}{\surd 2}\int_0^x \left\{\cosh\frac{x-\xi}{\surd 2}\sin\frac{x-\xi}{\surd 2} - \sinh\frac{x-\xi}{\surd 2}\cos\frac{x-\xi}{\surd 2}\right\} f(\xi)\, d\xi,$$

as may easily be proved by substitution. Thus we can get particular integral of $\dfrac{d^8u}{dx^8} - u = f(x)$ at once. For using symbolic notation

$$(D^8 - 1)\,u = f(x),$$

therefore
$$u = \tfrac{1}{2}\left\{\frac{1}{D^4 - 1} f(x) - \frac{1}{D^4 + 1} f(x)\right\}$$

$$= \tfrac{1}{4}\int_0^x \{\sinh(x-\xi) - \sin(x-\xi)\} f(\xi)\, d\xi$$

$$- \frac{1}{2\surd 2}\int_0^x \left\{\cosh\frac{x-\xi}{\surd 2}\sin\frac{x-\xi}{\surd 2} - \sinh\frac{x-\xi}{\surd 2}\cos\frac{x-\xi}{\surd 2}\right\} f(\xi)\, d\xi,$$

which may be simplified a little. The complementary function may be written down at once.

9426. (Professor SATIS CHANDRA RAY, M.A.)—Prove the identity

$$m\tan^{-1}y = n\tan^{-1}\left\{(-1)^{\frac{1}{2}}\frac{(y+i)^{m/n}+(y-i)^{m/n}}{(y+i)^{m/n}-(y-i)^{m/n}}\right\}, \text{ when } i=(-1)^{\frac{1}{2}}.$$

Solution by Professor MADHAVARAO, M.A.

Suppose $\tan^{-1}y = \phi$, and $m/n\,\phi = \theta$,

$$m\tan^{-1}y = n\,m/n\tan^{-1}y = n\theta = n\tan^{-1}(\tan\theta) = n\tan^{-1}\left\{i\,\frac{\sin\theta}{i\cos\theta}\right\},$$

but $$\frac{\sin\theta}{i\cos\theta} = \frac{-2i\sin\theta}{2\cos\theta} = \frac{\cos\theta - i\sin\theta-(\cos\theta+i\sin\theta)}{\cos\theta-i\sin\theta+(\cos\theta+i\sin\theta)}$$

$$= \frac{(\cos\phi-i\sin\phi)^{m/n}-(\cos\phi+i\sin\phi)^{m/n}}{(\cos\phi-i\sin\phi)^{m/n}+(\cos\phi+i\sin\phi)^{m/n}}$$

$$= \frac{(-i^2\cos\phi-i\sin\phi)^{m/n}-(-i^2\cos\phi+i\sin\phi)^{m/n}}{(-i^2\cos\phi-i\sin\phi)^{m/n}+(-i^2\cos\phi+i\sin\phi)^{m/n}} = \frac{(i+y)^{m/n}-(i-y)^{m/n}}{(i+y)^{m/n}+(i-y)^{m/n}},$$

therefore $$m\tan^{-1}y = n\tan^{-1}\left\{(-1)^{\frac{1}{2}}\frac{(i+y)^{m/n}-(i-y)^{m/n}}{(i+y)^{m/n}+(i-y)^{m/n}}\right\}.$$

9626. (Professor VUIBERT.)—Si a et b sont deux nombres entiers tels que la somme $a+b+1$ représente un nombre premier, démontrer que $1.2.3\ldots a\times 1.2.3\ldots b\pm 1 = \text{Mult.}\ (a+b+1)$, en prenant les signes $\pm$ suivant que a et b sont pairs ou impairs.

9625. (Professor COCHEZ, M.A.)—Si $2n+1$ est un nombre premier, démontrer que la somme des produits k et k' $(k<n)$ des carrés des n premiers nombres est divisible par $2n+1$.

Solution by Professor GENESE, M.A.; A. E. THOMAS; *and others.*

(9626). By Wilson's theorem $(a+b)!+1 = \text{M}\,(a+b+1)$, or, if

$$a+b+1=p,\quad a!\times(p-1)(p-2)\ldots(p-b)+1 = \text{M}\,(p),$$

i.e., $\text{M}\,(p)+(-1)^b a!\times b!+1 = \text{M}\,(p)$, $\therefore a!\times b!+(-1)^b = \text{M}\,(a+b+1)$.
In particular, if $a=b=n$, $(-1)^n(n!)^2+1 = \text{M}\,(2n+1)$.

(9625). Again, if x be prime to $2n+1$, one of the factors

$$(x^2-1)(x^2-2^2)(x^2-3^2)\ldots(x^2-n^2) \text{ is } \text{M}\,(2n+1).$$

Thus $$x^{2n}-s_1x^{2n-2}+s_2x^{2n-4}-\ldots(-1)^n(n!)^2 = \text{M}\,(2n+1);$$

by FERMAT, $x^{2n} = \text{M}\,(2n+1)+1$, therefore using 9626,

$$s_1x^{2n-2}-s_2x^{2n-4}+\ldots \text{ to } (n-1) \text{ terms} = \text{M}\,(2n+1).$$

Giving x values 1, 2, 3, ... $n-1$, we see that $s_k = \text{M}\,(2n+1)$ divided by a determinant equal to

$$1^2.2^2\ldots(n-1)^2(1^2-2^2)(1^2-3^2)\ldots\ldots\{(n-1)^2-n^2\},$$

and each of these factors is prime to $(2n+1)$.

9618. (Professor NEUBERG.)—Soient A′, B′, C′ les centres de gravité de trois masses m, n, p, appliquées, une première fois aux sommets A, B, C, une seconde fois aux sommets B, C, A, une troisième fois aux sommets C, A, B d'un triangle. Démontrer que les triangles ABC, A′B′C′ ont même angle de BROCARD.

Solution by Professor DE WACHTER.

Let O be any point in space, and put $OA = \alpha$, $OB = \beta$, $OC = \gamma$, $OA' = \rho_1$, $OB' = \rho_2$, $OC' = \rho_3$; then, we have

$$\rho_1 = \tfrac{1}{3}(m\alpha + n\beta + p\gamma); \quad \rho_2 = \tfrac{1}{3}(p\alpha + m\beta + n\gamma); \quad \rho_3 = \tfrac{1}{3}(n\alpha + p\beta + m\gamma).$$

Now, the Brocard-angle is given by its tangent $= 4S/(a^2+b^2+c^2)$; a, b, c, S, being the sides and area of ABC. But $4S/(a^2+b^2+c^2)$ is proportional to the vector-expression $V(\alpha\beta + \beta\gamma + \gamma\alpha) / \{(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2\}$

which $\equiv V(\rho_1\rho_2 + \rho_2\rho_3 + \rho_3\rho_1) / \{(\rho_1-\rho_2)^2 + (\rho_2-\rho_3)^2 + (\rho_3-\rho_1)^2\}$,

as appears from the above values of ρ_1, ρ_2, ρ_3. Hence the theorem.

8357. (Professor WOLSTENHOLME, M.A., Sc.D.)—Two conics S, S′ intersect in four points A, B, C, D; tangents to S, S′ at A meet CD in a, a'; tangents to S, S′ at B meet CD in b, b'; and tangents to S, S′ at C, D meet AB in c, c'; d, d' respectively; prove that

$$[Acc'B] = [Ad'dB] = [Caa'D] = [Cb'bD].$$

Solution by the PROPOSER.

Let λ be the anharmonic ratio of ACDB at any point on S; λ' at any point on S′; and let AB, CD meet in E. Consider the ranges which C [ACDD] and D [ACDB] make on AB; we get

$$\lambda = [AcEB] = \frac{Ac \,.\, EB}{AE \,.\, cB} = [AEdB] = \frac{AE \,.\, dB}{Ad \,.\, EB},$$

$$\lambda' = [Ac'EB] = \frac{Ac' \,.\, EB}{AE \,.\, c'B} = [AEd'B] = \frac{AE \,.\, d'B}{Ad' \,.\, EB};$$

whence $$\lambda/\lambda' = \frac{Ac \,.\, c'B}{Ac' \,.\, cB} = \frac{Ad' \,.\, dB}{Ad \,.\, d'B} = [Acc'B] = [Ad'dB].$$

So, taking the ranges of A [ACDB], B [ACDB] on CD,

$$\lambda = [aCDE] = \frac{aC \,.\, DE}{aD \,.\, CE} = [ECDb] = \frac{EC \,.\, Db}{ED \,.\, Cb},$$

$$\lambda' = [a'CDE] = \frac{a'C \,.\, DE}{a'D \,.\, CE} = [ECDb'] = \frac{EC \,.\, Db'}{ED \,.\, Cb'};$$

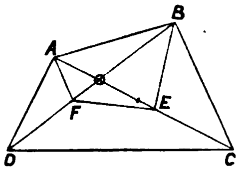
whence $$\lambda/\lambda' = \frac{Ca \,.\, a'D}{Ca' \,.\, aD} = \frac{Cb' \,.\, bD}{Cb \,.\, b'D} = [Caa'D] = [Cb'bD].$$

Hence, finally, $[Acc'B] = [Ad'dB] = [Caa'D] = [Cb'bD] = \lambda/\lambda'$.

[The theorem may be proved more briefly by the method of projections: say, projecting S, S′ into a confocal ellipse and hyperbola. The equal cross-ratios will all be harmonic if $\lambda + \lambda' = 0$, and it can easily be proved that in this case the invariant equation $\Theta\Theta' + \Delta\Delta = 0$ holds, so that the harmonic locus of S, S′ reduces to a line-pair; and the harmonic envelope to a point-pair.]

9681. (W. J. GREENSTREET, M.A.)—AC, BD are diagonals of a quadrilateral. AF parallel to BC cuts BD in F; BE parallel to AD cuts AC in E. Prove that EF is parallel to CD.

Solution by E. M. LANGLEY, M.A.; Rev. W. T. WELLACOTT, M.A.; *and others.*



Let AC, BD cut in O;

then OE : OB = OA : OD,

and OB : OC = OF : OA,

therefore OE : OC = OF : OD

hence EF is parallel to CD.

9734. (R. TUCKER, M.A.)—If we have

$$\lambda = \begin{vmatrix} 2b\cos C, & 2a\cos C, & c \\ 2c\cos B, & b, & 2a\cos B \\ a, & 2c\cos A, & 2b\cos A \end{vmatrix}, \quad \mu = \begin{vmatrix} c\cos C, & 2a\cos C, & c \\ b\cos B, & b & 2a\cos B \\ a\cos A, & 2c\cos A, & 2b\cos A \end{vmatrix},$$

prove that $3\mu = \lambda \cos A$.

Solution by Rev. D. THOMAS, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

Multiply λ by $\begin{vmatrix} \cos A & \cos B & \cos C \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$, or cos A, and the stated result is at once obtained.

9493. (Professor MATZ, M.A.)—If

$$(a^2+b^2+c^2)^2 = 3\,(a+b+c)\,(b+c-a)\,(c+a-l)\,(a+b-c),$$

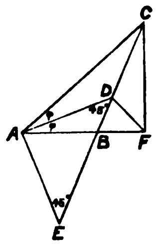
prove that a, b, c are all imaginary or all equal.

Solution by A. M. WILLIAMS, M.A., *and* Rev. J. L. KITCHIN, M.A.

The equation reduces to $(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2 = 0$. If a, b, c are real, each term must be 0, *i.e.*, $a = b = c$; otherwise the expression must take the form $X^2+Y^2+Z^2 = 0$, which is impossible unless a, b, c are all imaginary.

9695. (E. MIGNOT.)—Construire un triangle, connaissant un côté, le pied de la hauteur correspondante, et sachant que les bissectrices (intérieure et extérieure) d'un angle adjacent au côté donné sont égales.

Solution by Professor SCHOUTE.



When in ABC the bisectors AD, AE are equal, and F is the foot of the perpendicular from C on AB, D is the in-centre of ACF. For, if we put $\angle DAB = \phi$,

then $\quad \angle DCA = \angle BDA - \phi = 45° - \phi,$

$\angle FCD = 90° - \angle EBA = 45° - \phi$

thus $\quad \angle BFD = 45° = \angle BDA.$

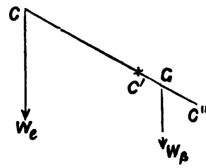
In the similar triangles ABD and ADF, we have

$$AB : AD = AD : AF, \quad \text{or} \quad AD^2 = AB.AF;$$

thus AD is to be found. Then the triangle ADE can be constructed, etc.

9421. (Professor HUDSON, M.A.)—A basin formed of a segment of a spherical surface is movable about a horizontal axis which coincides with a diameter of the base of the segment. Prove that the basin will upset if the ratio of the weight of the water poured in to the weight of the basin, is greater than the ratio of $d : D - 2d$, when d is the depth of the basin, D the diameter of the sphere from which it is cut.

Solution by Professors ABINASH BASÛ *and* IGNACIO BEYENS.



Take the section through the centre of the sphere, the centre of the plane face of the segment, the section being vertical. Let us consider the equilibrium of the segment. Let C be the centre of the sphere, C′ the centre of the plane face, G the centre of gravity of the segment, which is well known to be the mid-point of C′C″, where CC″ is the radius of the sphere. The fluid pressures give a resultant $= W_e$, the weight of the liquid passing through C. Taking moments about C′, we have $W_e\,(\frac{1}{2}D - d) = \frac{1}{2}W_\beta\,.\,d$. Whence the result follows.

9628. (Professor CURTIS, M.A.)—If X, X_1, X_2, X_3 are perpendiculars on any line from the in-centre and ex-centres of a triangle, prove that

$$X^{-1} = X_1^{-1} + X_2^{-1} + X_3^{-1}.$$

Solution by R. KNOWLES, B.A.; Prof. MATZ, M.A.; *and others.*

Let (l, m, n) be the given line, and

$$(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C)^{\frac{1}{2}} = d;$$

then X, X_1, X_2, X_3 are respectively equal to

$$(l+m+n)\,2\Delta/d\,(a+b+c),\quad (l+m+n)\,2\Delta/d\,(b+c-a),$$
$$(l+m+n)\,2\Delta/d\,(a+b-c),\quad (l+m+n)\,2\Delta/d\,(a+c-b).$$

Whence $$X^{-1} = X_1^{-1} + X_2^{-1} + X_3^{-1}.$$

9614. (Professor ABINASH BASU, M.A.)—Prove that, if

$$\phi(xy) \equiv b^2x^2 + a^2y^2 - a^2b^2 = 0$$

be the equation to a conic, and p and q be the lengths of the tangents from (x, y), then we shall have

$$p^2 + q^2 = 2\phi\left\{\phi(x^2+y^2) + a^4y^2 + b^4x^2\right\} / (\phi + a^2b^2)^2,$$
$$pq = \frac{\phi}{\phi + a^2b^2}\left\{(x^2+y^2)^2 + 2(a^2-b^2)(y^2-x^2) + (a^2-b^2)^2\right\}^{\frac{1}{2}}.$$

Solution by Rev. J. J. MILNE, M.A.; Prof. CHAKRAVARTI, M.A.; *and others.*

By *Companion to Weekly Problem Papers*, p. 268, § 8, the equation giving the lengths of the tangents is

$$t^4(a^2y^2 + b^2x^2)^2 - 2t^2\phi\left\{(a^2y^2 + b^2x^2)(x^2+y^2) + (a^2y^2 - b^2x^2)(a^2 - b^2)\right\}$$
$$+ \phi^2\left\{(x^2+y^2)^2 - 2(x^2-y^2)(a^2-b^2) + (a^2-b^2)^2\right\} = 0,$$

which at once gives the above results.

9643. (R. W. D. CHRISTIE.)—If $\Sigma_n^r = 1^r + 2^r + 3^r \ldots n^r$, prove that Σ_n^r is divisible by Σ_n^1.

Solution by R. KNOWLES, B.A.; J. C. ST. CLAIR, M.A.; *and others.*

By DE MORGAN'S *Calculus*, p. 257, we have

$$\Sigma_n^r = \tfrac{1}{2}n(n+1)\left\{\Delta 0^m + \tfrac{1}{3}\Delta^2 0^m (n-1) + \&c.\right\},$$

and this is divisible by $\Sigma_n^1 = 1 + 2 \ldots + n = \frac{1}{2}n(n+1)$.

9608. (S. TEBAY, B.A.)—Find the least heptagonal number which when increased by a given square shall be a square number.

Solution by R. W. D. CHRISTIE; *the* PROPOSER; *and others.*

The general form of heptagonal numbers is $\frac{1}{2}(5x^2 - 3x)$. Let a^2 be the given square, and assume

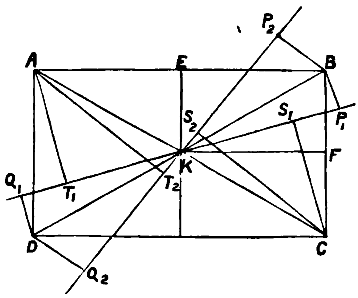
$$\tfrac{1}{2}(5x^2 - 3x) + a^2 = \left(a - \frac{m}{2n}x\right)^2,\quad \text{then}\quad x = \frac{2n(2am - 3n)}{m^2 - 10n^2}.$$

The convergent to $\sqrt{10}$ which makes $m^2-18n^2=1$ is $\frac{19}{6}$; hence $m=19$, and $n=6$, therefore $x=24\,(19a-9)$, which is the root of the heptagonal number required.

Take $a=1$, then $x=240$, and the heptagonal number is 143640. Thus $143641=379^2$.

3274 & 3825. (S. Watson.)—A line is drawn at random in direction but so as to cut a given rectangle; find the chance that it will intersect opposite sides, and show that, in the case of a square, the probability is ·4412712, or the odds are about 11 to 9 against.

Solution by D. Biddle.



Let the direction of the line be indicated by PQ, passing through K, the centroid of the rectangle. It may hold any position from KC to KA through B, or from KA to KC through D, but to regard it as lying between KF and KE will be sufficient, since the other quadrants are symmetrical with this. The actual line, parallel to PQ, may hold any position within a breadth of plane indicated by AT + CS; but in order to cut opposite sides, it must occupy the central portion indicated by BP + DQ. Now BP = DQ, and AT = CS; therefore BP/AT = the chance for any particular direction. Let PKC $=\theta$, BKC $=\alpha$. Then

$$BP_1/AT_1=\sin(\alpha-\theta)/\sin\theta, \text{ and } BP_2/AT_2=\sin(\theta-\alpha)/\sin\theta;$$

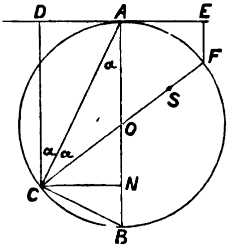
and we have the following integrals

$$\tfrac{1}{2}\pi\left(\int_{\frac{1}{2}\alpha}^{\alpha}\frac{\sin\alpha\cos\theta-\cos\alpha\sin\theta}{\sin\theta}\,d\theta+\int_{\alpha}^{\frac{1}{2}\pi+\frac{1}{2}\alpha}\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\sin\theta}\,d\theta\right)$$
$$=\frac{4\sin\alpha}{\pi}\log_e\frac{\sin\alpha}{\sin\frac{1}{2}\alpha}.$$

Consequently when the rectangle is a square, P = ·4412712, and the odds are 11 to 9 against.

9530. (E. Rutter.)—AB being the vertical diameter of a circle, a perfectly elastic ball descends down the chord AC, and is reflected by the plane BC; find the point where it will strike the circle after reflection.

Solution by Rev. J. L. KITCHIN, M.A.; *and* SARAH MARKS, B.Sc.



The velocity at C is that due to falling freely through AN. On reflection from CB the ball will rebound along CA, so that CA is a tangent to its parabolic path; DAE, horizontal through A, is the directrix to the parabola; and the diameter through O from C is equally inclined to AC with CD. Hence the focus is in this line at S, where CS = CD. Put r = radius of circle; then

$$CD = AN = r + r\cos 2\alpha = 2r\cos^2\alpha,$$

therefore $SF = 2r - 2r\cos^2\alpha = 2r\sin^2\alpha.$

Again, $EF = CD - 2r\cos 2\alpha = 2r(\cos^2\alpha - \cos 2\alpha) = 2r\sin^2\alpha$; $\therefore$ SF = FE. Hence F is a point on the parabola, and F is the point at which the particle strikes the circle.

[*Otherwise*:—CD + EF = 2AO; therefore CS + EF = CF = CS + SF; therefore EF = SF, &c.]

9529. (R. KNOWLES, B.A.)—The circle of curvature is drawn at a point P of a conic; M is the mid-point of their common chord; the diameter of the conic through M meets the normal at P in a point Q; prove that Q is the point through which pass all chords of the conic which subtend a right angle at P.

Solution by Prof. ABINASH BASU; Rev. T. GALLIERS; *and others.*

Taking P as the origin, and the tangents and normal as axes, let $ax^2 + by^2 + 2hxy + 2gx = 0$ be the equation to the conic; then the equations to the circle of curvature, the common chord, and the diameter are

$$(ax + hy + g)\,2h + (by + hx)(b - a) = 0;$$

$$b(x^2 + y^2) + 2gx = 0, \quad x(a - b) + 2hy = 0,$$

and we find that this diameter meets the normal at $[-2g(a+b), 0]$. Now this point is well known to be the point on the normal through which all the chords subtending a right angle at P pass. (See SALMON'S *Conics*, 6th Ed., page 175, Ex. 2.)

9655. (H. L. ORCHARD, M.A., B.Sc.)—Find the point of inflexion of the curve $\quad 3y^3 + x^3 + 7y^2x + 5x^2y + 11y^2 = 0.$

Solution by Professor SCHOUTE.

The curve has a cusp at the origin; therefore the line $y = mx$ cuts it in the point $\quad x = -\dfrac{11m^2}{(m+1)^2(3m+1)}, \quad y = -\dfrac{11m^3}{(m+1)^2(3m+1)},$

variable with the parameter m. The three points corresponding to m_1, m_2, m_3 are collinear under the condition

$$\begin{vmatrix} m_1^2, & m_1^3, & 3m_1^3+7m_1^2+5m_1+1 \\ m_2^2, & m_2^3, & 3m_2^3+7m_2^2+5m_2+1 \\ m_3^2, & m_3^3, & 3m_3^3+7m_3^2+5m_3+1 \end{vmatrix} = 0,$$

or $$5m_1m_2m_3 \begin{vmatrix} m_1, & m_1^2, & 1 \\ m_2, & m_2^2, & 1 \\ m_3, & m_3^2, & 1 \end{vmatrix} + \begin{vmatrix} m_1^2, & m_1^3, & 1 \\ m_2^2, & m_2^3, & 1 \\ m_3^2, & m_3^3, & 1 \end{vmatrix} = 0,$$

or $$(5m_1m_2m_3+m_2m_3+m_3m_1+m_1m_2)(m_2-m_3)(m_3-m_1)(m_1-m_2) = 0.$$

But the factors (m_2-m_3), (m_3-m_1), (m_1-m_2) being introduced improperly, the condition of collinearity is $5m_1m_2m_3+m_2m_3+m_3m_1+m_1m_2 = 0$, and this reduces to $5m^3+3m^2 = 0$ and for $m_1 = m_2 = m_3$. So the point of inflexion corresponds to the parameter $m = -\frac{3}{5}$, and has the coordinates

$$x = \tfrac{495}{16}, \quad y = -\tfrac{297}{16}.$$

9603. (J. Brill, M.A.)—If z be any complex function of x and y, and $w = f(z)$, prove that $\dfrac{dw}{dz} = \dfrac{\partial w}{\partial x}\Big/\dfrac{\partial z}{\partial x} = \dfrac{\partial w}{\partial y}\Big/\dfrac{\partial z}{\partial y}$.

Solution by the Proposer.

Let $w = f(z)$, where $z = \phi(x, y, i)$. Then we have

$$\frac{\partial w}{\partial x} = f'(z)\frac{\partial z}{\partial x} \text{ and } \frac{\partial w}{\partial y} = f'(z)\frac{\partial z}{\partial y}.$$

Therefore $$dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy = f(z)\left\{\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right\} = f'(z).dz.$$

Thus the value of the ratio $dw : dz$ is independent of that of the ratio $dy : dx$, and the symbol dw/dz has a definite meaning, viz.,

$$\frac{dw}{dz} = f'(z) = \frac{\partial w}{\partial x}\Big/\frac{\partial z}{\partial x} = \frac{\partial w}{\partial y}\Big/\frac{\partial z}{\partial y}.$$

9428. (W. P. Casey, C.E.)—Prove that, in Question 8755 [Vol. xlviii., p. 78], triangle A′B′C′ = 4 times triangle ABC + $\frac{1}{2}(a^2+b^2+c^2)$.

Solution by the Proposer; Professor Beyens; *and others.*

[See Fig. to Question 8755, Vol. xlviii., p. 78.] As a', b', and c' are the mid-points of the sides of ΔA′B′C′, and also the centres of the squares described on BC, AC, and AB respectively, therefore

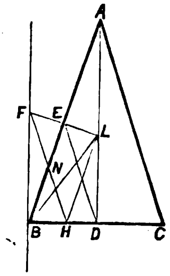
$$\Delta a'b'c' = \Delta ABC + \tfrac{1}{8}(a^2+b^2+c^2)$$

(Casey's *Sequel*, Ex. 12, Sec. viii., Book 6);

whence $$\Delta A'B'C' = 4\,\Delta ABC + \tfrac{1}{2}(a^2+b^2+c^2).$$

9564. (W. J. GREENSTREET, M.A.)—ABC is a triangle; AB = AC; D, E are mid-points of BC, AB. Join A, D; draw FEL perpendicular to AB cutting AD in L, and a perpendicular at B to BC in F; draw FH parallel to AC. Show HLF is a right angle.

Solution by the PROPOSER.



AD bisects the vertical angle A;

$F\widehat{B}E = \frac{1}{2}A$, because AD is parallel to BF;

$E\widehat{B}L = \frac{1}{2}A$, because L is circumcentre;

therefore FE = EL.

Also $B\widehat{F}N = \frac{1}{2}A$, $\because$ BF $\parallel$ AD, and HF $\parallel$ AC,

therefore BN = FN;

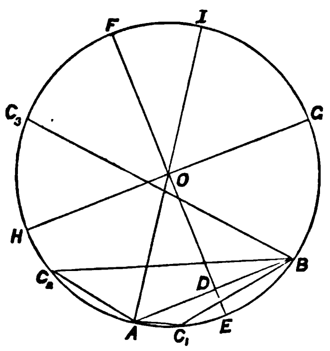
and NH $\parallel$ AC, $\therefore$ $N\widehat{H}B = C = B$, $\therefore$ NB = NH,

$\therefore$ N and E are mid-points of FH and FL, $\therefore$ EN $\parallel$ HL.

Therefore HLF is a right angle.

2458. (S. WATSON.)—Three points A, B, C are taken at random on the circumference of a given circle: show that the chance that a point P within the given circle, lies within the triangle ABC is $\pi^{-2}+\pi^{-3}$ $= \cdot 1335727$, or $\frac{2}{15}$ nearly.

Solution by D. BIDDLE.



The answer to this question is given by the average area of the triangle ABC, that of the circle being unity. A may be regarded as fixed; B, as anywhere on one semi-circumference determined by the diameter AI; C as anywhere on one semi-circumference determined by the diameter EF, which bisects AB. The other possible positions of B and C produce symmetrical results. C being in the semi-circumference EHF, may be in any one of the three arcs, AE, AH, HF. Let AOE = θ, COE = ϕ. Then we have the following integrals to give when reduced the area required:

$$\int_0^{\frac{1}{2}\pi}\int_0^{\theta} \sin\theta\,(\cos\phi-\cos\theta)\,d\theta\,.\,d\phi+\int_0^{\frac{1}{2}\pi}\int_{\theta} \sin\theta\,(\cos\theta-\cos\phi)\,d\theta\,.\,d\phi$$

$$=\int_0^{\frac{1}{2}\pi}(\sin^2\theta-\sin\theta\,.\,\cos\theta)\,d\theta+\int_0^{\frac{1}{2}\pi}(2\sin\theta\cos\theta+\sin^2\theta)\,d\theta=\tfrac{1}{2}\pi+\tfrac{1}{2},$$

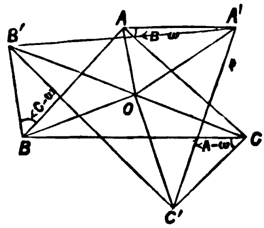
which must be divided (1) by $\frac{1}{4}\pi^2$, on account of $d\theta \,.\, d\phi$ (carried to $\frac{1}{2}\pi$ and π respectively), and (2) by π, the area of the circle with radius $= 1$. This gives the stated result.

9330. (J. O'Byrne Croke, M.A. Suggested by Question 9238.)—If lines AA′, BB′, CC′ of equal length be drawn in the same sense making angles ϕ, ϕ', ϕ'' with the sides BC, CA, AB, respectively, so that

$$\phi = \angle A - \omega, \quad \phi' = \angle B - \omega, \quad \phi'' = \angle C - \omega,$$

where ω is a constant angle, then $A'B' : B'C' : C'A' = AB : BC : CA$.

Solution by the Proposer.



The figure being constructed as indicated in the question, and O being the centre of the circumcircle, we have

$$\angle OAA' = \angle OAC + \angle B - \omega.$$

But $\quad 2\angle OAC + 2\angle B = 2$ right angles;

therefore $\quad \angle OAC + \angle B = 90°$,

therefore $\quad \angle OAA' = 90° - \omega = \angle OCC' = \angle OBB'$,

therefore $\quad OA' = OB' = OC'$.

Also, since $\quad \angle AOA' = \angle COC' = \angle BOB'$,

we have $\angle A'OC' = \angle AOC$, &c. Hence, the triangles A′B′C′, ABC are equiangular, and therefore $A'B' : B'C' : C'A' = AB : BC : CA$.

1769. (N'Importe.)—Solve the equation

$$\left\{(x+2)^2 + x^2\right\}^3 = 8x^4(x+2).$$

Solution by D. Biddle.

No real root of x in this equation can be positive; but a negative root is found in -1.

9612. (Professor Wolstenholme, M.A., Sc.D.)—A circle touches a given parabola in the point P, passes through the focus, and cuts the parabola again in Q, R; the two real common tangents to the circle and

parabola meet in T: prove that the straight lines PT, PQ, PR have all the same envelope, a tricusp quartic, one ousp real, $(\frac{1}{2}a, 0)$ if the equation of the parabola be $y^2 = 4a(x+a)$, and two impossible $(14a, \pm 3\sqrt{-1}a)$. There is a real bitangent $x = 2a$, its points of contact lying on the parabola.

Solution by the Proposer.

Let $a(m^2-1)$, $2am$ be the point P, [$y^2 = 4a(x+a)$ the given parabola], $x+a-my+am^2 = 0$ the tangent at P, $x+my+h = 0$ the chord QR; then for a certain value of λ, the equation

$$\lambda(y^2-4ax-4a^2)+(x+a-my+am^2)(x+my+h) = 0$$

must be a circle. This value is $1+m^2$, and since the circle passes through the focus (the origin), $h(1+m^2) = 4a\lambda$, or $h = 4a$ (so that QR meets the axis in a fixed point whose distance from the focus = lat. rect.), and the equation of the circle is

$$x^2+y^2-a(3m^2-1)x-a(3m-m^3)y = 0;$$

the centre is $\frac{1}{2}a(3m^2-1)$, $\frac{1}{2}a(3m-m^3)$, and the radius is $\frac{1}{2}a(1+m^2)^{\frac{3}{2}}$. Let $x+a-\mu y+a\mu^2=0$ be a common tangent; we shall have for μ the equation

$$[3m^2+1-\mu(3m-m^3)+2\mu^2]^2 = (1+\mu^2)(1+m^2)^3,$$

a quartic, having two roots m. Let μ_1, μ_2 be the two remaining roots; then $2m+\mu_1+\mu_2 = 3m-m^3$, and $m^2\mu_1\mu_2 = \frac{1}{4}[(3m^2+1)^2-(1+m^2)^3]$ $=\frac{1}{4}(3m^2+6m^4-m^6)$, so that $\mu_1+\mu_2=m-m^3$, $\mu_1\mu_2=\frac{1}{4}(3+6m^2-m^4)$. Hence the equation of PT will be $4mx+y(3-m^2) = 2am(1+m^2)$. Now let $a(m_1^2-1)$, $2am_1$ be the coordinates of Q, then $m_1^2+2mm_1+3 = 0$; and the equation of PQ, being $x+a-y(m+m_1)+amm_1=0$, becomes, on eliminating m, $4m_1x+y(3-m_1^2) = 2am_1(1+m_1^2)$, only differing from that of PT by the substitution of m_1 for m. The envelope is easily found to be a tricusp quartic

$$(x-14a)^{-\frac{1}{2}}+(4x+3iy-2a)^{-\frac{1}{2}}+(4x-3iy-2a)^{-\frac{1}{2}} = 0;$$

or $$(X+Y+7Z)^{-\frac{1}{2}}+(Y+Z+7X)^{-\frac{1}{2}}+(Z+X+7Y)^{-\frac{1}{2}} = 0,$$

where $X = x+iy$, $Y = x-iy$, $Z = -4a$; the equation of the parabola being $X^{\frac{1}{2}}+X^{\frac{1}{2}}+Z^{\frac{1}{2}} = 0$, and that of the circle $p^3YZ+q^3ZX+r^3XY = 0$, where $p+q+r = 0$, the coordinates of the point of contact P being $p^2 : q^2 : r^2$.

Of course the generalised property is obvious. A given conic S is inscribed in a given triangle ABC, another conic S′ is drawn circumscribing ABC, touching S in a point P and cutting S in Q, R; the common tangents to S, S′ meet in T: prove that the straight lines PT, PQ, PR have for common envelope a tricusp quartic. The invariant relations between two such conics are $\Theta^2 = 4\Theta\Delta'$, $2\Theta\Theta' = 27\Delta\Delta'$, the cubic for λ when $\lambda S+S'$ breaks into factors being always reducible to $2z^3-12z^2+9z-4=0$.

In the particular case of the circle and parabola, it follows from PT, PR having the same envelope, that if a circle be drawn through the focus touching the parabola in Q, the real common tangents to this circle and the parabola will intersect in a point lying on QP, and similarly for R.

9394. (Professor SCHOUTE.)—To find in point-coordinates or in plane-coordinates the equation of the locus of the line that meets three lines p', p'', p''' the line-coordinates p_k', p_k'', p_k''' ($k = 1, 2, 3, 4, 5, 6$) of which are given.

Solution by the PROPOSER.

When y_1, y_2, y_3, y_4 and z_1, z_2, z_3, z_4 are the coordinates of the points y, z, the coordinates of the line (y, z) and the identic relation between them are

$$p_1 = y_1z_4 - y_4z_1, \qquad p_4 = y_2z_3 - y_3z_1,$$
$$p_2 = y_2z_4 - y_4z_2, \qquad p_5 = y_3z_1 - y_1z_3,$$
$$p_3 = y_3z_4 - y_4z_3, \qquad p_6 = y_1z_2 - y_2z_1,$$
$$p_1p_4 + p_2p_5 + p_3p_6 = 0 \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

When x_1, x_2, x_3, x_4 are the current coordinates, the equations of the planes through the line (y, z) and one of the vertices of the tetrahedron of reference are found by equalising to zero three of the four quantities t_i in the determinant

$$\begin{vmatrix} x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \\ t_1, & t_2, & t_3, & t_4 \end{vmatrix} = 0;$$

thus

$$\left.\begin{array}{l} S_1 \equiv \qquad\qquad p_3x_2 - p_2x_3 + p_4x_4 = 0 \\ S_2 \equiv -p_3x_1 \qquad\quad + p_1x_3 + p_5x_4 = 0 \\ S_3 \equiv \ p_2x_1 - p_1x_2 \qquad\quad + p_6x_4 = 0 \\ S_4 \equiv -p_4x_1 - p_5x_2 - p_6x_3 \qquad\quad = 0 \end{array}\right\} \ldots\ldots\ldots\ldots(2).$$

In the same manner we may represent the three given lines p', p'', p''' in point-coordinates by the equations

$$\left.\begin{array}{l} S_1' = 0 \\ S_2' = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} S_1'' = 0 \\ S_2'' = 0 \end{array}\right\}, \qquad \left.\begin{array}{l} S_1''' = 0 \\ S_2''' = 0 \end{array}\right\}.$$

Now the intersection of the planes

$$S_1' + kS_2' = 0, \quad S_1'' + lS_2'' = 0 \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

belongs to the locus in question, when they cut the line p''' in the same point. By elimination of x_1, x_2, x_3, x_4, the equations

$$S_1' + kS_2' = 0, \quad S_1'' + lS_2'' = 0, \quad S_1''' = 0, \quad S_2''' = 0$$

give for the relation between k and l, and for the result of the elimination of k and l between this relation and the two equations (3), *i.e.*, the required equation,

$$\begin{vmatrix} kp_3', & p_3', & kp_1' - p_2', & kp_5' + p_4' \\ lp_3'', & p_3'', & lp_1'' - p_2'', & lp_5'' + p_4'' \\ 0, & p_3''', & -p_2''', & p_4''' \\ p_3''', & 0, & p_1''', & p_5''' \end{vmatrix} = 0, \qquad \begin{vmatrix} S_1', & S_2', & S_3', & S_4' \\ S_1'', & S_2'', & S_3'', & S_4'' \\ 0, & p_3''', & -p_2''', & p_4''' \\ -p_3''', & 0, & p_1''', & p_5''' \end{vmatrix} = 0$$

$$\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Evidently this equation corresponds to the hyperboloid (p', p'', p'''). First, the equations (2) show that the quantities S are linear in the current coordinates, therefore (4) represents a surface of the second order. This surface contains the lines p' and p'', the quantities S' disappearing for any point of p', and the quantities S'' for any point of p''. And by adding the different vertical rows after they have been multiplied successively by x_1, x_2, x_3, x_4 one obtains two cyphers followed by S_1''' and S_2'''; so the left member of (4) multiplied by one of the current coordinates disappears for any point of p''', &c.

When we use plane-coordinates instead of point-coordinates we obtain an entirely analogous equation. And both equations may be written into 18 different forms, by changing the order of the lines p', p'', p''' and the coordinates.

Finally, the hyperboloid (p', p'', p''') may also be represented by line-coordinates, by the equation of the complex formed by the tangents of the hyperboloid. This equation is found in the following way.

The six quantities $p_k = \lambda p_k' + \mu p_k'' + \nu p_k'''$ represent line-coordinates under the condition (1), or

$$\mu\nu \sum_1^6 p_k'' p_{k\pm 3}''' + \nu\lambda \sum_1^6 p_k''' p_{k\pm 3}' + \lambda\mu \sum_1^6 p_k' p_{k\pm 3}'' = 0 \ldots\ldots\ldots\ldots (5).$$

The corresponding lines p form a surface of the second order. For the condition that an arbitrary line q meets a line p is

$$\sum_1^6 p_k q_{k\pm 3} = 0, \text{ or } \lambda \sum_1^6 p_k' q_{k\pm 3} + \mu \sum_1^6 p_k'' q_{k\pm 3} + \nu \sum_1^6 p_k''' q_{k\pm 3} = 0 \ldots (6),$$

and this equation and the preceding determine two sets of proportions $\lambda : \mu : \nu$. Now, the line q touches the hyperboloid (p', p'', p'''), when these two sets coincide. Now, we contemplate λ, μ, ν as homogeneous coordinates of a point in a plane; then the condition is that the line (6) must touch the conic (5), and this condition is after Hesse's bordered determinant

$$\begin{vmatrix} 0, & \sum_1^6 p_k' p_{k\pm 3}'', & \sum_1^6 p_k''' p_{k\pm 3}', & \sum_1^6 p_k' q_{k\pm 3} \\ \sum_1^6 p_k' p_{k\pm 3}'', & 0, & \sum_1^6 p_k'' p_{k\pm 3}''', & \sum_1^6 p_k'' q_{k\pm 3} \\ \sum_1^6 p_k''' p_{k\pm 3}', & \sum_1^6 p_k'' p_{k\pm 3}''', & 0, & \sum_1^6 p_k''' q_{k\pm 3} \\ \sum_1^6 p_k' q_{k\pm 3}, & \sum_1^6 p_k'' q_{k\pm 3}, & \sum_1^6 p_k''' q_{k\pm 3}, & 0 \end{vmatrix} = 0.$$

About this equation compare *Mathematische Annalen*, von CLEBSCH, V., page 283.

8792. (Professor WOLSTENHOLME, M.A., Sc.D.)—If a, b, c be three conterminous edges of any tetrahedron, x, y, z the opposite edges, V the volume expressed in terms of a, b, c, x, y, z, prove that, if A be the dihedral angle opposite a, $dV/da = \frac{1}{6}ax \cot A$.

Solution by the PROPOSER.

In the tetrahedron OABC, let the lengths of the edges OA, OB, OC be a, b, c; those of the respectively opposite edges x, y, z; and let the dihedral angles opposite these respectively be A, B, C; X, Y, Z; also let the angles at C, ACO, OCB, BOA be α, β, γ.

Then $V = \frac{1}{6}xyc\,(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma)^{\frac{1}{2}}$,

also $\cos\alpha = \dfrac{y^2+c^2-a^2}{2yc}$, $\cos\beta = \dfrac{c^2+x^2-b^2}{2cx}$, $\cos\gamma = \dfrac{x^2+y^2-z^2}{2xy}$;

whence
$$144V^2 = 4x^2y^2z^2 - x^2(y^2+c^2-a^2)^2 - y^2(c^2+x^2-b^2)^2 - c^2(x^2+y^2-z^2)^2 + (y^2+c^2-a^2)(c^2+x^2-b^2)(x^2+y^2-z^2);$$

$$288V\,\partial V/\partial a = 2a\left\{2x^2(y^2+c^2-a^2)-(c^2+x^2-b^2)(x^2+y^2-z^2)\right\}$$
$$= 4ayc\,x^2(\cos\alpha-\cos\beta\cos\gamma) = 4ax\,.\,xyc\sin\beta\sin\gamma\cos A;$$

also $6V = p_1 \times 2\Delta ABC$, p_1 being the perpendicular from O on ABC,
$= c\sin\beta\sin A \times xy\sin\gamma = xyc\sin\beta\sin\gamma\sin A$;

whence
$$6\,\partial V/\partial a = ax\cot A.$$

[Hence when b, c, y, z and $a+x$ are given, for the maximum volume,
$$\partial V/\partial a\,.\,da + \partial V/\partial x\,.\,dc = 0,$$
also $da+dx=0$, so that $\partial V/\partial a = \partial V/\partial x$, whence $\cot A = \cot X$, or $A = X$. Similarly, if b, c, y, z, and $a-x$ be given, the volume will be a maximum when $A+X = 180°$; if a, x, $b+y$, $c-z$ be given, the volume will be a maximum when $B=Y$, and $C+Z = 180°$; if a, x, $b+y$, $c+z$ be given, the conditions for a maximum volume are $B=Y$, $C=Z$ (and therefore $b=y$, $c=z$); if $a+x$, $b+y$, $c+z$ be given, for a maximum volume, the tetrahedron will be equifacial, and if $2p$, $2q$, $2r$ be the given sums, the maximum volume will be $\left[2(q^2+r^2-p^2)(r^2+p^2-q^2)(p^2+q^2-r^2)\right]^{\frac{1}{2}}$. It would seem that if $a+x$, $b-y$, $c-z$ be given, the volume would be a maximum when $A=X$, and $B+Y = C+Z = 180°$, but I rather think these conditions cannot be satisfied in a finite tetrahedron. So, when $a-x$, $b-y$, $c-z$ are given, the conditions for a maximum being $A+X = 180°$, $B+Y = 180°$, $C+Z = 180°$, the tetrahedron will have the lengths of its edges infinite. In this last case, it is obvious we can make each edge as large as we choose, and that there can be no true maximum volume. I should like to see a satisfactory investigation, in the cases where (1) a, x, $b-y$, $c-z$, (2) $a+x$, $b-y$, $c-z$ are given whether there can be a maximum volume.]

9589. (R. TUCKER, M.A.)—PQ, QR are normals to a parabola at P, Q; determine when the coordinates of R are minimum coordinates. If the tangents at P, R intersect in T, and TN is an ordinate, prove that it passes through the orthocentre (O) of PQR, and that TO cuts PR on the axis; find also the tangents of the angles P, Q, R.

Solution by R. Knowles, B.A.; Rev. T. Galliers, M.A.; *and others.*

Let h, k be the coordinates of the pole of PQ; x_1, y_1, &c., those of PQR; then

$$y_1 = -4a^2/k, \quad y_2 = 2(k^2+2a^2)/k;$$
$$y_3 = -\left\{4a^2k/(k^2+2a^2) + 2(k^2+2a^2)/k\right\},$$

and this latter expression is a minimum when $k = 2^{\frac{1}{2}}a$. The coordinates of P, Q, R, T then are

$$-2^{\frac{3}{2}}a,\ 2a;\ 2^{\frac{5}{2}}a,\ 8a;\ -5\,.\,2^{\frac{1}{2}}a,\ \tfrac{25}{2}a;\ 5a,\ -7a/2^{\frac{1}{2}};$$

and the equations to PQ, QR, PR are respectively

$$y = 2^{\frac{1}{2}}(x-4a);\ y = -2^{\frac{3}{2}}(x-10a);\ 7y = -2^{\frac{3}{2}}(x-5a);$$

and the coordinates of O, the intersection of perpendiculars from P, Q, R, are $5a$, $-5a/2^{\frac{3}{2}}$; therefore the equation to TO is $x = 5a$, which is also that of TN; therefore TN passes through O, and TR meets PR on the axis; the tangents of P, Q, R are respectively $-3\,.\,2^{\frac{1}{2}}$; $-2^{\frac{1}{2}}$; $2^{\frac{1}{2}}/5$.

8999. (S. Tebay, B.A.) — Adopting the usual notation, other expressions for the volume of a tetrahedron are

$$V = \tfrac{1}{3}\,.\,\frac{\Delta_1^2+\Delta_2^2+\Delta_3^2-\Delta_4^2}{a\cot X + b\cot Y + c\cot Z}$$
$$= \tfrac{1}{3}\,.\,\frac{\Delta_1^2+\Delta_2^2+\Delta_3^2+\Delta_4^2}{a\cot X + b\cot Y + c\cot Z + x\cot A + y\cot B + z\cot C}.$$

Solution by the Proposer; Professor Beyens; *and others.*

Projecting every three of the faces on the fourth, we have

$\Delta_2\cos Z + \Delta_3\cos Y + \Delta_4\cos A = \Delta_1$, $\Delta\cos X + \Delta_1\cos Z + \Delta_4\cos B = \Delta_2$...(1, 2),

$\Delta_1\cos Y + \Delta_2\cos X + \Delta_4\cos C = \Delta_3$, $\Delta_1\cos A + \Delta_2\cos B + \Delta_3\cos C = \Delta_4$...(3, 4).

Multiply (4) by Δ_4, thus

$$\Delta_1\Delta_4\cos A + \Delta_2\Delta_4\cos B + \Delta_3\Delta_4\cos C = \Delta_4^2 \quad\text{.................(5).}$$

Again, multiply (1) by Δ_1, (2) by Δ_2, (3) by Δ_3, and add, taking account of (5); then

$$\Delta_1^2+\Delta_2^2+\Delta_3^2-2\Delta_2\Delta_3\cos X - 2\Delta_3\Delta_1\cos Y - 2\Delta_1\Delta_2\cos Z = \Delta_4^2 \quad\text{......(6).}$$

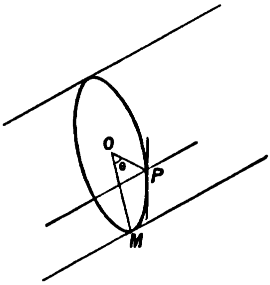
Now, $\quad 2\Delta_2\Delta_3\cos X = 2\Delta_2/a\sin X\,.\,\Delta_3\,.\,a\cot X = 3V\,a\cot X,$

and similarly for the other terms. Thus (6) gives the first value of V; and similar expressions are obtained by considering the other solid angles, which immediately lead to the second value of V.

9368. (G. H. Bryan, B.A.)—A rough hollow circular cylinder (coefficient μ, inner radius a) having its axis inclined to the horizon at

an angle α is made to rotate with angular velocity v/a; show that it is possible for a particle to move down the cylinder with uniform velocity $v\left(\dfrac{\mu^2+1}{\mu^2\cot^2\alpha-1}\right)^{\frac12}$, provided $\mu > \tan\alpha$.

Solution by the PROPOSER; SARAH MARKS, B.Sc.; *and others.*



Let the particle P be moving with uniform velocity u down the direction of the generator at P, O centre, and M lowest point of circular section at P, MOP $= \theta$.

Since friction is in direction opposite to that of slipping, its components along generator and tangent to circular section at

P are $\qquad \dfrac{\mu R u}{(u^2+v^2)^{\frac12}},\ \dfrac{\mu R v}{(u^2+v^2)^{\frac12}}.$

Hence, since acceleration of particle $= 0$, we get by resolving along radius (OP), generator, and tangent to circular section,

$$g\cos\alpha\cos\theta = R,$$

$$g\sin\alpha = \mu R u/(u^2+v^2)^{\frac12},\quad g\cos\alpha\sin\theta = \mu R v/(u^2+v^2)^{\frac12}.$$

Eliminating θ and g/R, we get

$$\cot^2\alpha = \frac{u^2+v^2+\mu^2v^2}{\mu^2u^2};\ \text{therefore}\ \mu^2\cot^2\alpha-1 = (1+\mu^2)\frac{v^2}{u^2};$$

therefore $$u = v\left(\frac{1+\mu^2}{\mu^2\cot^2\alpha-1}\right)^{\frac12}.$$

This is only real if $\mu > \tan\alpha$.

8845. (S. TEBAY, B.A.)—A beam, a inches wide, and b inches thick, can be cut into an exact number of boards m/n^{ths} of an inch thick, whether the boards be a inches wide, or b inches wide. Find the width of the saw-gate and the number of boards in each case. If $a > b$, which method would you adopt in practice?

Solution by the PROPOSER.

Let x be the number of saw-gates when a is divided, and s the width of each. The waste from saw-gates $= sx$, and the remainder $a-sx$ divided by m/n gives the number of boards; that is, $n/m\,(a-sx) = x+1$. Therefore $s = na-m(x+1)/nx$. Similarly, when b is divided, $s = nb-m(y+1)/ny$; y being the number of saw-gates in this case. From these two values of s, we have $x/y = (na-m)/(nb-m)$. Take $x = t\,(na-m)$, and $y = t\,(nb-m)$, t being a convenient numerical factor. Then $s = (1-mt)/nt$.

The number of boards in the two cases is

$$t\,(na+m)+1 \quad\text{and}\quad t\,(nb-m)+1;$$

and the respective quantities of boarding

$$bl/12\,\{t\,(na-m)+1\},\quad al/12\,\{t\,(nb-m)+1\}.$$

If $a > b$, the former expression will be greater than the latter if $tm > 1$.

Example.—If $a = 20$, $b = 13$, $s = \frac{1}{8}$ inch, $m/n = \frac{3}{4}$ inch, we find $t = \frac{2}{7}$, and the number of boards 13 inches wide $= 23$, and the number 20 inches wide $= 15$. Thus, if l be the length of the beam, the respective quantities of boarding are $\frac{299}{12}l$ and $\frac{300}{12}l$; the difference being $l/12$ feet. In this case it appears to be more advantageous to cut the boards 20 inches, and this agrees with the above limitation.

9560. (J. O'Byrne Croke, M.A.)—An elliptical lamina with its conjugate axis horizontal, and plane inclined to the vertical, falls under the influence of gravity, and in its fall suffers contraction along the transverse axis, so that its orthogonal projection on the horizontal plane through the lower and fixed extremity of that axis is always a circle of radius r; determine the motion of the foci along the axis, and show that their paths in the vertical plane are given by the equation

$$x^2/y^2 = \left\{x^2 + (r \pm y)^2\right\}/r^2.$$

Solution by the Proposer.

Suppose the centre of the lamina to fall freely under the influence of gravity, the weight being concentrated thereat; let θ, θ', be the inclinations of the transverse axis to the vertical, initially and at the time t, respectively; and let x be the distance one of the foci has moved from its original position at the time t; then corresponding to the distance x travelled by the focus towards the centre, the latter point has fallen vertically through a space $= \frac{1}{2}gt^2$. The initial focal distance

$$= \left(\frac{r^2}{\sin^2\theta} - r^2\right)^{\frac{1}{2}} = r\cot\theta,$$

and focal distance at time $t = r\cot\theta'$; therefore $x = r\,(\cot\theta - \cot\theta') = \frac{1}{2}gt^2$.

Again, let $\rho =$ radius vector from fixed end of transverse axis as origin; $2a =$ length of that axis at the time t; and $f =$ focal distance at that moment, and we have $\rho = a \pm f = a \pm r\cot\theta' = r\,\dfrac{1 \pm \cos\theta'}{\sin\theta'}$, the polar equation of loci; which for locus of the upper focus gives $\rho = r\cot\frac{1}{2}\theta'$, and for that of the lower, $\rho = r\tan\frac{1}{2}\theta'$. Employing rectangular coordinates, we have for locus of upper focus,

$$\cot\tfrac{1}{2}\theta' = \frac{x}{y}, \text{ and } \rho^2 = x^2 + (x+y)^2, \text{ and therefore } \frac{x^2}{y^2} = \frac{x^2 + (r+y)^2}{r^2};$$

while for locus of lower focus, we have

$$\tan\tfrac{1}{2}\theta' = \frac{x}{y}, \text{ and } \rho^2 = x^2 + (r-y)^2, \text{ whence } \frac{x^2}{y^2} = \frac{x^2 + (r-y)^2}{r^2};$$

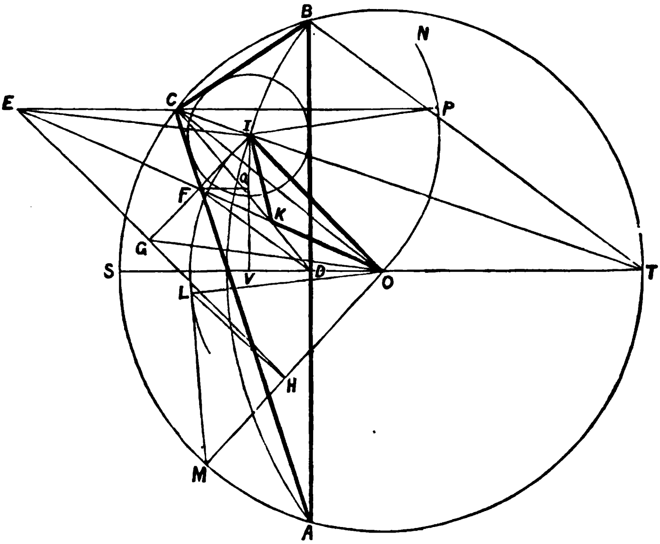
or, writing both in one equation, $x^2/y^2 = \left\{x^2 + (r \pm y)^2\right\}/r^2$.

[An interesting property of these curves, which may be seen at once by constructing a figure, is that the intercept they make on any straight line through the lower extremity of the transverse axis of the ellipse subtends a right angle at their common origin.]

4117. (The Editor.)—**Construct a plane triangle, having given the three distances apart of its centroid, incentre, and circumcentre.**

Solution by D. Biddle.

Let O, I, K be the circumcentre, incentre, and centroid respectively, of the required triangle; and let OK = k, OI = l. Produce OK to E, making OE = $3k$, and bisect OE in F. Join IF, and produce the line to



G, making IG = 2IF. Draw OH parallel and equal to IG, and HL at right angles to it to meet the arc drawn from O as centre with radius l. Produce OH, and draw LM at right angles to OL. Then OM = R, the radius of the circumcircle, and MH = $2r$, or twice the radius of the in-circle. These circles can at once be described. But although the triangle ABC can readily be drawn, as hereafter shown, with sufficient accuracy for practical purposes, the methods of Euclid must be departed from to a slight extent, as they have hitherto been also in trisecting an angle. Thus it can easily be effected by linkage; I, O being fixed points, let OC, IP be jointed to a straight rod at C and P respectively, points separated by a distance = R. Then, as P moves along the arc ON, the rod of which CP forms a portion will cover E, a point already given; and at the same time indicate C, one apex of the required triangle, after which A and B are found without further difficulty. But a simpler method is to draw the arc ON from centre I with radius l, and, having a ruler marked at two points separated by the distance R, adjust it so that (P) on the last-mentioned arc, and (C) on the circumcircle, shall coincide with these marks, whilst E is also in the same straight line.

In order, however, to comprehend the foregoing construction, an analysis is required. Let AB be the given base of a triangle of which the circumcentre is O. ST, drawn perpendicular to AB through O, bisects AB in D. Let $OD = x$. Then the arc BIA, drawn from T as centre with radius TB, will be the locus of the incentre, whilst the apex C lies on ASB; for $\angle BTS = \frac{1}{2}A + \frac{1}{2}B$. Moreover, if from C a straight line be drawn parallel to ST, and $CE = 2x$, then OE passes through the centroid K, and $OE = 3OK$, for K lies on CD, and $CK = 2KD$. Moreover, $FD = \frac{1}{4}OC = \frac{1}{4}R$, since $FK = \frac{1}{4}OK$. Now, let $\angle COS = \theta$, and $CTS = \frac{1}{2}\theta$. We have $TB = (2R^2 + 2R \cdot OD)^{\frac{1}{2}}$, $IV = TB \sin\frac{1}{2}\theta$, $OV = TB\cos\frac{1}{2}\theta - R$,

$$QV = \tfrac{1}{2}R\sin\theta, \quad FQ = \tfrac{1}{2}R\cos\theta + OD - OV;$$

whence

$$IF = R\left\{3\tfrac{1}{4} + \frac{4 \cdot OD}{R} - \sin\theta\sin\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}} + \frac{OD^2}{R^2} + \cos\theta\left(1 + \frac{OD}{R}\right) - 2\cos\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}} - \cos\theta\cos\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}} - \frac{2 \cdot OD}{R}\cos\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}},$$

which reduces to

$$IF = R\left\{3\tfrac{1}{4} + \frac{4 \cdot OD}{R} + \frac{OD^2}{R^2} - \left(3 + \frac{2 \cdot OD}{R}\right)\cos\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}} + \cos\theta\left(1 + \frac{OD}{R}\right)\right\}^{\frac{1}{2}}.$$

Now, by consideration of the triangle ITO, it is evident that

$$l^2 = OI^2 = R^2\left\{3 + \frac{2 \cdot OD}{R} - 2\cos\tfrac{1}{2}\theta\left(2 + \frac{2 \cdot OD}{R}\right)^{\frac{1}{2}}\right\}.$$

But $\cos\theta = 2\cos^2\frac{1}{2}\theta - 1$, therefore $IF = \frac{1}{2}l^2/R$, whence it follows that $2IF : l = l : R$. This gives us the circumcircle as defined by the radius OM. The incircle is further given by the well-known formula, $l = (R^2 - 2Rr)^{\frac{1}{2}}$, whence $2r = (R^2 - l^2)/R$; and we have the diameter of the incircle given in MH. In order to find OD ($= x$), or EC ($= 2x$), we consider that EC is parallel to OD, and that CI bisects $\angle PCO$; wherefore we take the two triangles ECI, ICO, and observe that

$$\cos ECI = -\cos PCI = -\cos ICO.$$

Moreover, all chords drawn through I to the circumcircle are divided at that point into two portions, the product of which is constant, namely, $R^2 - OI^2$, and this $= 2rR$; therefore $CI = 2rR/TI = 2rR/(2R^2 + 2Rx)^{\frac{1}{2}}$. Taking R as unity, and $EI = m$, which is given, we have

$$\cos ECI = \left(4x^2 + \frac{4r^2}{2 + 2x} - m^2\right) \Big/ \frac{8rx}{(2 + 2x)^{\frac{1}{2}}},$$

$$\cos ICO = \left(1 + \frac{4r^2}{2 + 2x} - l^2\right) \Big/ \frac{4r}{(2 + 2x)^{\frac{1}{2}}}.$$

Consequently, since $\cos ICO = -\cos ECI$, we obtain the cubic equation

$$(2x)^3 + (3 - l^2)(2x)^2 + (4r - 2m^2 + 4r^2)(2x) - (2m^2 - 4r^2) = 0.$$

This, of course, would enable us to find the approximate value of OD or EC in any given case. But, for practical geometrical purposes, the methods of construction previously given are superior, and the proof of

them is so simple as scarcely to need putting in detail. Suffice it to say, CI bisects $\angle$ BCA, and therefore cuts the circumcircle at T (the mid-point of the arc BTA, and the extremity of that diameter which is perpendicular to the base AB). But CP, in line with EC, is parallel to ST, as already proved. Therefore $\angle$ PCI = ITO = ICO; and since, in the triangles ICO, ICP, we have CI common and IP = OI, therefore CP = CO.

[Mr. Carr remarks that, instead of proving the construction of R and r *ab initio*, as above, Mr. Biddle might have availed himself of the fact that E is the orthocentre of the triangle, and F the centre of the nine-point circle (of radius $= \frac{1}{2}$R). This circle touches the incircle, therefore IF $= \frac{1}{2}R - r$, which, taken in conjunction with the equation referred to above, viz., $l = (R^2 - 2Rr)^{\frac{1}{2}}$, gives $2IF = l^2/R$.]

8183. (D. Edwardes.)—If $A+B+C = \pi$, prove that

$$\frac{\sin 2A + \sin 2B}{2\cos 2C - 1} + \frac{\sin 2B + \sin 2C}{2\cos 2A - 1} + \frac{\sin 2C + \sin 2A}{2\cos 2B - 1}$$
$$= 4\left(\frac{\sin 2A + \sin 2B}{2\cos 2C - 1}\right)\left(\frac{\sin 2B + \sin 2C}{2\cos 2A - 1}\right)\left(\frac{\sin 2C + \sin 2A}{2\cos 2B - 1}\right).$$

Solution by G. G. Storr, M.A.

Adding the last two quantities on the left-hand side, we get as numerator

$$\sin 4A + \sin 4B + 2\sin 2C(\cos 2A + \cos 2B) - 2\sin 2C - (\sin 2A + \sin 2B);$$

hence the left-hand side becomes

$$(\sin 2A + \sin 2B)\left\{\frac{1}{2\cos 2C - 1} - \frac{1 + 8\cos C \sin A \sin B}{(2\cos 2A - 1)(2\cos 2B - 1)}\right\},$$

and the numerator of the bracketed part of this expression reduces to

$$4(\sin 2B + \sin 2C)(\sin 2C + \sin 2A),$$

whence the required result follows.

9682. (R. Knowles, B.A. Suggested by Question 9544.)—G is the point of intersection of the diagonals of a quadrilateral inscribed in a circle; PHX, a diameter through G, meets the third diagonal EF in X; PX is produced to K, so that KX = HX. Prove that the points E, K, F, P are concyclic.

Solution by E. M. Langley, M.A.; Prof. Matz, M.A.; *and others.*

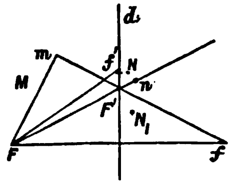
It is known that the triangle EGF is self conjugate with respect to the circle, and therefore that, if C be the centre, G is the orthocentre of CEF.

Hence EX . XF = CX . XG = square of tangent from X,
= PX . XH = PX . XK;

therefore E, K, F, P are concyclic.

9457. (Professor NASH, M.A.)—An indefinite number of ellipses are drawn with an endless string of length $2a$; one focus is fixed, and the other moves on a given straight line; find the envelope of the ellipses.

Solution by Professor NEUBERG.



Soient E et E′ deux des ellipses de la question, F le foyer commun, F′ et f' les seconds foyers situés sur la droite donnée d, M un de leurs points d'intersection. Par hypothèse,

$$(1)\quad FF'+MF+MF' = 2a,$$
$$Ff'+MF+Mf' = 2a,$$

d'où $\quad FF'+MF' = Ff'+Mf'.$

Les points F′ et f' sont donc sur une même ellipse dont F et M sont les foyers. Laissons l'ellipse E fixe et faisons tendre f' vers F′ sur la droite d; M tend vers une position limite m appartenant à l'enveloppe cherchée, et d sera tangente à une ellipse passant par F′ et ayant pour foyers F, m. Donc d est la bissectrice extérieure de l'angle FF′m, et F′m passe par le point f symétrique de F par rapport à d. L'égalité (1) donne maintenant $2a = fF'+mF+mF = mF+mf$. Le lieu de m est donc une ellipse E ayant pour foyers F, f, et pour grand axe $2a$.

Les ellipses E, E′ peuvent se couper en un point N, situé à droite de d. La droite NF passe nécessairement entre F et F′; car on ne pourrait avoir

$$FF'+N_1F+N_1F' = Ff'+N_1F+N_1f'.$$

Lorsque E′ tend vers E, la limite n de N est sur la droite FF′, et par suite est un sommet de E, et comme $nF = a$, le lieu de n est une circle γ.

Il n'y a que la demi-ellipse E à gauche de d, et l'arc de la circonférence γ à droite de d qui répondent à l'énoncé. La demi-ellipse située à droite de d est l'enveloppe des ellipses ayant pour foyers F et un point F′ mobile sur d, la différence entre le grand axe et la distance des foyers étant égale à $2n$; à la même enveloppe appartient aussi la partie de γ située à gauche d'une parallèle menée par F à d. La partie restante de γ correspond à des hyperboles.

[This is a repetition of Question 4432, two solutions of which are given on pp. 33, 34 of Vol. XXII., where however, in both cases, the condition that the string should be *endless* has been overlooked. The same error is repeated in TAYLOR's *Conics*, where the problem appears as Question 339 on p. 130. It may be easily proved geometrically that the envelope consists of a circle and an ellipse.]

9711. (Professor MANNHEIM.)—Du pôle d'une normale en M à une ellipse donnée on abaisse une perpendiculaire sur le diamètre qui passe par ce point : cette droite rencontre en P ce diamètre, en Q la normale en M à l'ellipse, et en R la perpendiculaire abaissée du centre de l'ellipse sur la tangente en M à cette courbe. On demande les lieux décrits par P, Q, R, et l'enveloppe de la droite PQR, lorsque M parcourt l'ellipse donnée.

Solution by A. Provost, Lic.-ès.-Sc.; R. Knowles, B.A.; *and others.*

The coordinates of the point M being $(a\cos\phi,\ b\sin\phi)$, the equation of the line PQR is $\quad x\,.\,a\cos\phi + y\,.\,b\sin\phi - (a^2+b^2) = 0$;
whence, if O be the centre, $\text{OM}\,.\,\text{OP} = a^2+b^2$, and also $= \text{OK}\,.\,\text{OR}$, where K is the foot of the perpendicular OR, on the tangent at M.

Consequently, the locus of the point P is the inverse curve of the ellipse relatively to its director circle; the locus of the point R is the reciprocal polar ellipse of the given ellipse relatively to the same circle, and is also the envelope of the straight line PQR. The locus of the point Q will be determined by the preceding equation and that of the normal

$$x\,.\,a\sin\phi - y\,.\,b\cos\phi - (a^2-b^2)\sin\phi\cos\phi = 0,$$

that is, by
$$ax = \left[a^2+b^2+(a^2-b^2)\sin^2\phi\right]\cos\phi,$$
$$by = \left[a^2+b^2-(a^2-b^2)\cos^2\phi\right]\sin\phi.$$

This curve is contained between the loci of the points P and R.

[Professor Mannheim remarks that "Les auteurs des solutions auraient pu remarquer que MQ est égal au rayon de courbure de l'ellipse en M."]

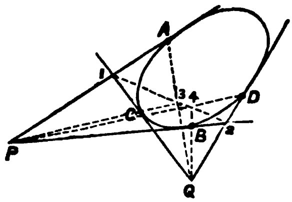
9633. (Professor Chakravarti, M.A.)—A triangle circumscribes an ellipse. Two of its vertices move on confocal ellipses; prove that the third vertex-locus is another confocal.

Solution by W. J. Greenstreet, M.A.; Sarah Marks, B.Sc.; *and others.*

Using Darboux's notation, let the conic be $\text{U} = y^2-4xz = 0 = (\rho-\rho_1)^2$, and let $f(\rho,\ \rho_1)=0,\ f'(\rho,\ \rho_1)=0$ be the conics on which the vertices move. If $\rho,\ \rho_1,\ \rho_2$ be the parameters of the three tangents to U which are the sides of the triangle, the coordinates of the vertices are $(\rho,\ \rho_1)$, $(\rho,\ \rho_2)$, $(\rho_1,\ \rho_2)$; the second and third, the conics $f,\ f'$. Therefore we have $f(\rho,\ \rho_2)=0,\ f'(\rho_1,\ \rho_2)=0$. Eliminating ρ_2 between these gives the locus required. The notation is briefly this:—We get all tangents to $\text{U}=y^2-4xz=0$ by varying μ in $\mu^2x+\mu y+z = 0$, and if the tangent pass through $(x',\ y',\ z')$, we get μ from the equation $\mu^2x'+\mu y'+z' = 0$; taking ρ and ρ_1 as the roots of this, $\dfrac{1}{x'} = -\dfrac{y'}{\rho+\rho_1} = \dfrac{z'}{\rho\rho_1}$, and we can take $\rho,\ \rho_1$ as coordinates of $(x',\ y',\ z')$.

9721. (Rev. T. C. Simmons, M.A.)—From any two points P, Q, tangents PA, PB, QC, QD are drawn to a given conic. Prove that the four intersections of PA with QC, PB with QD, PC with QA, and PD with QB are collinear.

Solution by A. PROVOST, Lic.-ès-Sc.; Professor SCHOUTE; *and others.*



Let 1, 2, 3, 4 be the intersections of PA, QC, &c. We may consider at A and C respectively two tangents A1, AP, and C1, CQ, and so obtain a BRIANCHON'S hexagon P2QC1A, where the straight lines PC, 21, QA have a common point which is 3. And so for the point 4.

9452. (Professor BORDAGE.)—Prove that $u=(x^2+y^2+z^2)^{\frac{1}{2}}$ satisfies the equation $\dfrac{d^4u}{dx^4}+\dfrac{d^4u}{dy^4}+\dfrac{d^4u}{dz^4}+2\dfrac{d^4u}{dy^2dz^2}+2\dfrac{d^4u}{dz^2dx^2}+2\dfrac{d^4u}{dx^2dy^2}=0.$

Solution by FREDERIC R. J. HERVEY.

The equation may be written

$$(d^2/dx^2+d^2/dy^2+d^2/dz^2)^2\,u=0.$$

Now, $u=(x^2+y^2+z^2)^{\frac{1}{2}}$ gives $du/dx=x/u$, $d^2u/dx^2=(u^2-x^2)/u^3$;

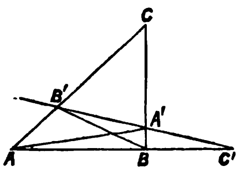
therefore $(d^2/dx^2+d^2/dy^2+d^2/dz^2)\,u=2/u=v$, suppose.

Again, $dv/dx=-2x/u^3$, $d^2v/dx^2=-2\,(u^2-3x^2)/u^5$;

therefore $(d^2/dx^2+d^2/dy^2+d^2/dz^2)\,v=0.$

8819. (R. KNOWLES, B.A.)—Prove that the mid-points of the three diagonals of a complete quadrilateral are collinear.

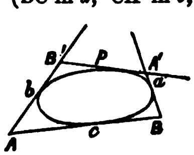
Solution by Professor WOLSTENHOLME, M.A., Sc.D.



The following proof (from statical principles) of this well-known theorem (said to be due to NEWTON) is, I think, not as well-known as it deserves to be. Let A′, B′, C′ be the points where any straight line meets the sides of a triangle ABC, and consider the resultant of forces represented by A′B, A′B′; AB, AB′. The resultant of A′B, A′B′ passes through the mid-point of BB′, so also does that of AB, AB′; hence the resultant of the four forces does so. The

resultant of A′B′, AB′ passes through the mid-point of AA′, so also does the resultant of AB, A′B; hence the resultant of the four forces does so. Moreover, the force A′B is equivalent to the two CB, A′C, and similarly for the others; or the four are equivalent to AC, AC′; A′C, A′C′; CB, C′B; CA, C′A; and the resultant of each of these pairs passes through the mid-point of CC′; hence the resultant of the four passes through the mid-points of all three diagonals, which must therefore be collinear.

Moreover, the straight line on which these mid-points lie is the locus of the centres of all conics inscribed in the quadrilateral; for, if a conic be inscribed touching them in the points a, b, c, p (BC in a, CA in b, AB in c, and A′B′C′ in p); the resultant of forces A′p, A′a passes through the centre; so also does that of pB′, bB′; that of Ab, Ac; and that of Bc, Ba; and these forces together are equivalent to the forces A′B, A′B′; AB, AB′; hence the resultant of these four forces passes through the centre of any inscribed conic. (An inscribed ellipse has been supposed in the figure, but the corresponding proof is obvious for any other case.) Hence the locus of the centres of all inscribed conics is a straight line which bisects the three diagonals (which are themselves particular cases of inscribed conics, the mid-point of each being centre).



[This proof was given in the *Quarterly Journal* more than twenty years ago, but has never been reproduced in any elementary book, as it really might well have been.]

9622. (Professor CATALAN.)—Parmi tous les quadrilatères convexes dont les angles et le périmètre sont donnés, quel est le plus grand en surface?

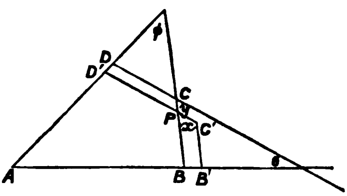
Solution by Professor GENESE, M.A.

Let ABCD be the maximum, AB′C′D′ a consecutive quadrilateral with common angle A, and let BC, C′D′ meet at P, C′P $= x$, PC $= y$. Then, by FERMAT's principle,

area DCPD′ = area PBB′C′,

PD′ $y \sin$ P = PB $x \sin$ P,

or CD y = CB x ......(1).



Again, since the perimeter is constant, we have

$$\text{BB}' + (\text{B}'\text{C}' - \text{BC}) + (\text{C}'\text{D}' - \text{CD}) - \text{DD}' = 0,$$

$$x\frac{\sin \text{P}}{\sin \text{B}} - \left(y + \frac{x \sin\theta}{\sin \text{B}}\right) + \left(x + \frac{y\sin\phi}{\sin \text{D}}\right) - y\frac{\sin \text{P}}{\sin \text{D}} = 0,$$

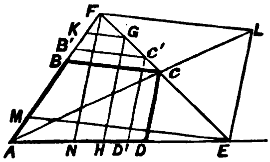
$$\frac{x}{\sin \text{B}}(\sin \text{B} + \sin \text{P} - \sin\theta) = \frac{y}{\sin \text{D}}(\sin \text{D} + \sin \text{P} - \sin\phi) \text{......(2)},$$

where θ is the angle (AB, DC), ϕ the angle (BC, AD). From (1) and (2),

$$CD\frac{\sin B+\sin P-\sin\theta}{\sin B}=BC\frac{\sin D+\sin P-\sin\phi}{\sin D}.$$

This condition expresses the fact that the circle touching AB, BC, CD is equal to that touching BC, CD, DA; and since equal circles inscribed in the same angle C must coincide, we learn that the required quadrilateral must circumscribe a circle.

[The Rev. J. G. BIRCH solves the Question thus:—It is easy to construct a triangle of given angles, and of given perimeter. Let, then, FAE be one of the given angles of the required quadrilateral, and ANF and AME two others; and let the perimeter of each of the triangles ANF and AME be equal to the given perimeter of the required quadrilateral. Join EF by a straight line. Through any point G of that line draw KG and GH parallel to ME and NF. Then it can be readily seen that the perimeter of the quadrilateral GKAH is equal to the perimeter of each of the triangles ANF and AME, *i.e.*, to the perimeter of the required quadrilateral. It remains to find the greatest of the quadrilaterals thus constructed. Through F draw FL parallel to ME, and through E draw EL parallel to NF. Join L to A, cutting FE in C. Through C draw BC and CD parallel to ME and NF. Then ABCD is the greatest quadrilateral. For, if C′ be a point infinitely near to C, it is only at the point C that the complementary areas C′B and CD′ become equal in the limit. This may be seen as follows:—Let P be intersection of B′C′ and CD produced, and Q that of BC and C′D′. Then at C, and nowhere else, the diagonal PQ of the elementary parallelogram QC′PC coincides in direction with that of parallelogram whose adjacent sides are PB′ and PD or in the limit BC and CD, because C′P : CP = FL : LE = BC : CD. Hence C′B and CD are ultimately equivalent to the complementary parallelograms about diagonal and therefore equal. Hence ABCD is required quadrilateral.



The centres of the circles inscribed in the triangles formed by the lines DA, AB, and BC, and by the lines BA, AD, and CD, lie on the bisector of angle A, and the trigonometrical formula at once shows at some distance from A. Hence ABCD circumscribes a circle.]

9415. (E. W. REES, B.A.)—I is the incentre of a triangle ABC, E_1, E_2, E_3 are the three ex-centres; prove that, if A′ be the mid-point of IE, &c., (1) A′, B′, C′ lie on the circumcircle, (2) $B'C' = 2R\cos\frac{1}{2}A$, (3) $\Delta = 8\Delta'\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

Solution by the PROPOSER.

If we construct the nine-point circle of $E_1E_2E_3$, I is the orthocentre, and IA, IB, IC its perpendiculars. Three of the points on this N. P. C. which is the circumcircle of ABC, are the mid-points of IO_1, IO_2, IO_3, which are the points A′, B′, C′.

9684. (E. RUTTER.)—Prove that the intersections of the perpendiculars of the four component triangles of every complete quadrilateral range in the same right line.

Solution by E. M. LANGLEY, M.A.; BELLE EASTON, B.Sc.; *and others.*

Let O be the common point of intersection of the four circumcircles of the triangles; then the Simson lines of the four triangles with respect to O are one and the same straight line. But this passes through the mid-points of the line joining O to each orthocentre; therefore the four orthocentres are in the same straight line.

9513. (W. J. C. SHARP, M.A.)—If tangents be drawn from a given point to a curve (class m), the other tangents to the curve from the points of contact of these last will all touch a curve of class $m-2$.

Solution by Professor MACMAHON.

If, of the n^2 intersections of two n-ics, np lie on a p-ic, the remaining $n(n-p)$ will lie on an $(n-p)$-ic. Replacing now the p-ic by two coincident lines, and the second n-ic by the n tangents at the n intersections, we learn that, if tangents be drawn at the intersections of a given line with an n-ic, the other intersections of these tangents will lie on an $(n-2)$-ic. The reciprocal of this is the theorem proposed.

9079. (Professor HUDSON, M.A.)—A person who can work up to a tenth of a horse-power draws a bucket, mass M lbs., up a well by means of a wheel and axle, exerting a constant force equal to the weight of F lbs.; if a be the radius of the wheel, b of the axle, prove (1) that the man cannot go on longer than $55\text{M}b^2/\{\text{F}ag\,(\text{F}a-\text{M}b)\}$ seconds (neglecting the mass of the machine); and (2) account for the apparently wrong dimensions of this result.

Solution by the PROPOSER; RAJ MOHAN SEN; *and others.*

1. $\text{F}g$ is the force applied by the man, therefore $\text{F}ag/b$ is the tension in the rope drawing up the bucket, since the mass of the machine is neglected; hence $\text{F}ag/\text{M}b-g$ is the acceleration of the bucket, and $(\text{F}a-\text{M}b)\,gt/\text{M}b$ is the velocity of the bucket at time t; thus

$$a(\text{F}a-\text{M}b)\,gt/(\text{M}b^2)$$

is the velocity of the end of the radius a at time t, and

$$\text{F}ag\,(\text{F}a-\text{M}b)\,gt/(\text{M}b^2)$$

is the rate at which the man is working. He can go on till this is 55, therefore $$t = \frac{55\text{M}b^2}{\text{F}ag\,(\text{F}a - \text{M}b)}.$$

2. The number 55 in this result, being a rate of working, represents a quantity of dimensions ML^2T^{-3}.

9405. (R. Knowles, M.A.)—PQ is a chord of a rectangular hyperbola normal at P; the diameter through Q meets the curve in R. Prove that PR is the chord of curvature at P.

Solution by Professors Abinash Basû, Ignacio Beyens, *and others.*

Take P as origin, and the tangent at P as the axis of x, and the normal the axis of y. The equations to the hyperbola and the osculating circle are $\quad x^2 - y^2 + 2\lambda_1 xy + 2\lambda y = 0, \quad x^2 + y^2 + 2\lambda y = 0;$

hence $y - \lambda_1 x = 0$ is the chord of curvature. The equation to QR is therefore $$y^{-1}\left\{x \,.\, \lambda_1^{-1}(y - \lambda_1 x) + x^2 - y^2 + 2\lambda_1 xy + 2\lambda y\right\} = 0,$$

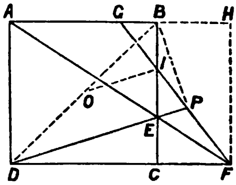
or $$x(\lambda_1^{-1} + 2\lambda_1) - y + 2\lambda = 0.$$

The centre is the intersection of the lines $$x + \lambda_1 y = 0, \quad -2y + 2\lambda_1 x + 2\lambda = 0 \;\ldots\ldots\ldots\ldots\ldots(2,\,3).$$

And it is easy to see that (1) passes through the intersection of (2) and (3) in the centre for $(2)/\lambda_1 + (3) = (1)$.

9715. (Professor Morel.)—Par l'un des sommets A d'un carré ABCD on mène une droite quelconque coupant en E le côté BC et en F le côté CD. Démontrer que la droite qui joint F au milieu I de BE touche le cercle inscrit au carré et rencontre la droite ED sur le cercle circonscrit à ce carré.

Solution by Professor Schoute; G. G. Morrice, M.B.; *and others.*



The points ABGH are harmonic as projections of EB, I, and the point at infinity on BC from F.

$\therefore \quad (\text{AG} + \text{BH})^2 = \text{AB}^2 + \text{GH}^2 = \text{GF}^2,$

or $\quad \text{AG} + \text{BH} = \text{GF},$

that is, $\quad \text{AG} + \text{DF} = \text{AD} + \text{GF};$

which proves that FG touches the in-circle of the square.

Therefore, when O is the centre of the square, OI bisects the angle EIG. And as DE is parallel to OI, the triangle IEP is isosceles. So I is the centre of the circle through B, E, P, and the angle BPE or BPD is right; which proves that P is on the circumcircle of the square.

9165. (Professor BORDAGE.)—If a triangle having a constant angle is deformed in such a manner that, the summit of the constant angle being fixed and the opposite side passing through a fixed point, one of the two other summits describes a straight line, prove that the third summit describes *a conic.*

Solution by R. KNOWLES, B.A.

Let A be the constant angle; D the fixed point through which BC passes; $y = mx + c$ the equation to the given line; take A for the origin of rectangular coordinates, and AD for an axis; let the coordinates of D, B, C be $(a, 0)$, (x', y'), (x, y) respectively, and let $\tan\theta = y'/x'$. Since xy is on AC and on CD, we have

$$y/x = \tan(\mathrm{A}+\theta) = (x' \tan \mathrm{A} + y') / (x' - y' \tan \mathrm{A}) \quad\ldots\ldots\ldots\ldots(1),$$

$$y/(a-x) = y'/(a-x') \text{ and } y' = mx' + c \quad\ldots\ldots\ldots\ldots\ldots(2, 3).$$

Eliminating x', y' from (1), (2), (3), we find the locus of C to be

$$c \tan \mathrm{A}\, x^2 + (c \tan \mathrm{A} + am + a \tan \mathrm{A})\, xy - a(1 + m \tan \mathrm{A})\, y^2$$
$$- ac \tan \mathrm{A}\, x + acy = 0,$$

a conic passing through A and D.

9536. (Professor ABINASH CHANDRA BASÚ, M.A.)—Find the equations of the focal lines of the cone $a/x + b/y + c/z = 0$.

Solution by D. EDWARDES; SARAH MARKS, B.Sc.; *and others.*

The focal lines are perpendicular to the cyclic planes of the reciprocal cone whose equation is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0.$$

Comparing this equation with

$$\lambda(x^2+y^2+z^2) + (lx+my+nz)(l'x+m'y+n'z) = 0,$$

we have $\lambda + ll' = a^2, \quad \lambda + mm' = b^2, \quad \lambda + nn' = c^2,$

whence $b^2n^2 + c^2m^2 = \lambda(m^2+n^2) + mn(mn'+m'n).$

Also $m'n + mn' = -2bc$, &c., therefore $(bn+cm)^2 = \lambda(m^2+n^2)$; hence

$$\frac{(bn+cm)^2}{m^2+n^2} = \frac{(cl+an)^2}{n^2+l^2} = \frac{(am+bl)^2}{l^2+m^2},$$

and the required equations are

$$\frac{(bz+cy)^2}{y^2+z^2} = \frac{(az+cx)^2}{z^2+x^2} = \frac{(ay+bx)^2}{x^2+y^2}.$$

7725. (EDITOR.)—If $A_1B_1C_1$ be a plane triangle of area Δ_1 and sides a, b, c, and through the angles A_1, B_1, C_1, three straight lines be drawn making equal angles θ with the sides A_1B_1, B_1C_1, C_1A_1; prove that (1) if Δ_2 be the area of the triangle $A_2B_2C_2$ thus formed; R_1, R_2, R_3, R_4 the circum-radii of the triangles $A_1B_1C_1$, $A_1A_2C_1$, $A_1B_1B_2$, $B_1C_1C_2$, and δ the area of the triangle formed by joining the circumcentres of the last three triangles; then the following relations will subsist:—

$$(\alpha)\quad \Delta_2 - \Delta_1 = \sin^2\theta\,\frac{a^4+b^4+c^4}{8\Delta_1} + \sin 2\theta\,\frac{a^2+b^2+c^2}{4};$$

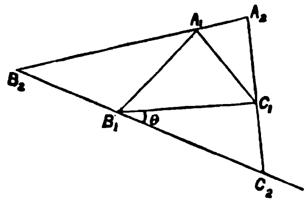
$$(\beta)\quad \delta\Delta_1 = (\tfrac{1}{4}ab)^2 + (\tfrac{1}{4}bc)^2 + (\tfrac{1}{4}ca)^2; \quad (\gamma)\quad R_1 = (R_2R_3R_4)^{\frac{1}{3}};$$

(2) if the operation be repeated, we shall have

$$\Delta_m/\Delta_n = \sin^{2(m-n)}\theta\,\{\cot\theta + \Sigma(\cot A_1)\}^{2(m-n)};$$

and (3) Δ_m, Δ_n will have their sides parallel to each other, if $\kappa\pi/\theta$ be an integer $= m \sim n$, κ being 0 or any positive integer; and the two triangles will be in a position of relative inversion or similarity, according as κ is odd or even.

Solution by Professors MUKHOPADHYAY, M.A., and IGNACIO BEYENS.



1. Since Δ_1 and Δ_2 are similar, we have

$$\Delta_2 - \Delta_1 = \Delta A_1B_1B_2 + \Delta B_1C_1C_2 + \Delta C_1A_1A_2$$

$$= \frac{c^2\sin\theta\sin(B_1+\theta)}{2\sin B_1} + \frac{a^2\sin\theta\sin(C_1+\theta)}{2\sin C_1} + \frac{b^2\sin\theta\sin(A_1+\theta)}{2\sin A_1}$$

$$= \tfrac{1}{2}\sin^2\theta\,(c^2\cot B_1 + a^2\cot C_1 + b^2\cot A_1) + \tfrac{1}{4}\sin 2\theta\,(a^2+b^2+c^2).$$

But
$$\cot B_1 = \frac{2bc\cos B_1}{2bc\sin B_1} = \frac{c^2+a^2-b^2}{4\Delta_1},\ \&c.,$$

therefore
$$\Delta_2 - \Delta_1 = \sin^2\theta\,\frac{a^4+b^4+c^4}{8\Delta_1} + \sin 2\theta\,\frac{a^2+b^2+c^2}{4}\ \ldots\ldots\ldots\ldots\ldots(1),$$

or
$$\frac{\Delta_2}{\Delta_1} = 1 + \sin^2\theta\,\frac{a^4+b^4+c^4}{8\Delta_1^2} + \sin 2\theta\,\frac{a^2+b^2+c^2}{4\Delta_1}$$

$$= \sin^2\theta\left\{\cot^2\theta + 2\cot\theta\,\frac{a^2+b^2+c^2}{4\Delta_1} + \frac{(a^2+b^2+c^2)^2}{16\Delta_1^2}\right\}$$

$$= \sin^2\theta\left\{\cot\theta + \frac{a^2+b^2+c^2}{4\Delta_1}\right\}^2.$$

2. Now
$$\frac{a^2+b^2+c^2}{4\Delta_1} = \frac{\sin A_1}{2\sin B_1\sin C_1} + \frac{\sin B_1}{2\sin C_1\sin A_1} + \frac{\sin C_1}{2\sin A_1\sin B_1}$$
$$= \cot A_1 + \cot B_1 + \cot C_1.$$

Therefore $\Delta_2/\Delta_1 = \sin^2\theta\,\{\cot\theta + \Sigma\cot A_1\}^2 = \Delta_3/\Delta_2 = \Delta_4/\Delta_3 = \ldots = \kappa$,

therefore
$$\frac{\Delta_m}{\Delta_n} = \ldots = \frac{\kappa^{m-1}\Delta_1}{\kappa^{n-1}\Delta_1} = \kappa^{m-n}$$
$$= \sin^{2(m-n)}\theta\,\{\cot\theta + \Sigma(\cot A_1)\}^{2(m-n)}.$$

3. The sides of Δ_m will be inclined to those of Δ_1 at an angle $(m-1)\theta$, respectively. Therefore, if Δ_m and Δ_n have their sides parallel,

$$(m-1)\theta \sim (n-1)\theta = \kappa\pi,$$

where κ is any positive integer; or $\kappa\pi/\theta = m \sim n$. If κ is even, the sides $B_m C_m$ and $B_n C_n$ will be in the same direction, and the triangles will be in a position of relative similarity; if κ is odd, the sides will be parallel, but opposite in direction.

(α) $R_1 = \dfrac{a}{2\sin A_1} = \dfrac{b}{2\sin B_1} = \dfrac{c}{2\sin C_1}$, $\quad R_2 = \dfrac{b}{2\sin A_1} = \dfrac{b}{a} R_1$,

$R_3 = \dfrac{c}{2\sin B_1} = \dfrac{c}{b} R_1$, $\quad R_4 = \dfrac{a}{2\sin C_1} = \dfrac{a}{c} R_1$;

therefore $\quad R_2 R_3 R_4 = b/a \,.\, c/b \,.\, a/c \,.\, R_1^3 = R_1^3$;

therefore $\quad R_1 = (R_2 R_3 R_4)^{\frac{1}{3}}$.

(β) Let O_1, O_2, O_3, O_4 be the centres of the four circles, then

$$O_1O_2 = R_1 \cos B + R_2 \cos A = R_1/a\,(a\cos B + b\cos A) = c/a \,.\, R_1.$$

Similarly, $\quad O_1O_3 = a/b \,.\, R_1$, and $O_1O_4 = b/c \,.\, R_1$,

$$\delta = \tfrac{1}{2} O_1O_2 \,.\, O_1O_3 \sin A_1 + \tfrac{1}{2} O_1O_3 \,.\, O_1O_4 \sin B_1 + \tfrac{1}{2} O_1O_4 \,.\, O_4O_2 \sin C_1$$
$$= \tfrac{1}{2} R_1^2 \left\{ c/b \sin A + a/c \sin B + b/a \sin C \right\} = \tfrac{1}{4} R_1 \left\{ ac/b + ab/c + bc/a \right\},$$

$$\Delta_1 = \frac{abc}{4R_1}; \quad \text{therefore} \quad \delta\Delta_1 = (\tfrac{1}{4}ac)^2 + (\tfrac{1}{4}ab)^2 + (\tfrac{1}{4}bc)^2.$$

9556. (A. Russell, B.A.)—If a, b, c, d, e are the lengths of the sides of a pentagon in which and about which circles can be drawn, prove (1) that

$$a\,\frac{c-d}{b+e-a} + b\,\frac{d-e}{c+a-b} + c\,\frac{e-a}{d+b-c} + d\,\frac{a-b}{e+c-d} + e\,\frac{b-c}{a+d-e} = 0;$$

and (2) express this condition in the form of a determinant.

Solution by the Proposer.

If Δ be the area of the pentagon, we have

$$\tan \tfrac{1}{2}A = \frac{\Delta}{s(s-b-d)}, \quad \tan \tfrac{1}{2}D = \frac{\Delta}{s(s-e-b)}.$$

Thus $\quad \sin A - \sin D = \dfrac{d-e}{c+a-b} \sin(A+D)$,

similarly $\quad \sin B - \sin E = \dfrac{e-a}{d+b-c} \sin(B+E)$, &c.;

therefore $\quad \dfrac{c-d}{b+e-a} \sin(E+C) + \ldots = 0.$

But, since a circle can be described about the pentagon,

$$\frac{a}{\sin(E+C)} = \frac{b}{\sin(A+D)} = \ldots = -2R.$$

Thus $a\dfrac{c-d}{b+e-a}+\ldots=0$. To express this in the form of a determinant, let $A_1=c-d$, $A_2=d-e$, $A_3=e-a$, &c. Then

$$A_1+A_2+A_3+A_4+A_5=0,\quad aA_1+bA_2+cA_3+dA_4+eA_5=0,$$

$$(c+d)\,A_1+(d+e)\,A_2+\ldots=0,\quad \frac{1}{cd}A_1+\frac{1}{de}A_2+\ldots=0,$$

$$\frac{a}{b+e-a}A_1+\frac{b}{c+a-b}A_2+\ldots=0.$$

Eliminating A_1, A_2, A_3, A_4, A_5, we get

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ a & b & c & d & e \\ c+d & d+e & e+a & a+b & b+c \\ \frac{1}{cd} & \frac{1}{de} & \frac{1}{ea} & \frac{1}{ab} & \frac{1}{bc} \\ \frac{a}{b+e-a} & \frac{b}{c+a-b} & \frac{c}{d+b-c} & \frac{d}{e+c-d} & \frac{e}{a+d-e} \end{vmatrix}=0.$$

9257. (F. R. J. Hervey.)—Tangents to a hypocycloid, in number equal to the class of the curve, meet at a point, and make with a fixed line angles whose sum $=\phi$; show that $\tan\phi$ is independent of the position of the point of concurrence, and state the corresponding theorem for the epicycloid.

Solution by the Proposer.

These properties will be most briefly proved by referring to Quest. 4323 [Vol. xxxi., p. 25]. If two points P, Q describe a circle with velocities always in the ratio $m:n$ (m, n being positive whole numbers prime to each other), the envelope of PQ is an epicycloid or hypocycloid, of class $m+n$, according as the points move in the same or opposite senses. Proceeding as in the solution referred to, we obtain, for the two cases respectively, the following equations; taking centre for origin, unit radius, and OX passing through a point of coincidence of P, Q, or vertex of the curve; (x, y) is the point of concurrence,

angle made by PQ with $\text{OX}=\frac{1}{2}\left\{(n\pm m)\,\theta+\pi\right\}$, $t\equiv\cos\theta+i\sin\theta$,

$$(x-iy)\,t^{n+m}-t^n-t^m+x+iy=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$t^{n+m}-(x-iy)\,t^n-(x+iy)\,t^m+1=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In (2) we have product of roots $=\pm 1$, and sum of corresponding values of θ an even or odd multiple of π; in either case $\phi=k\pi$, where k

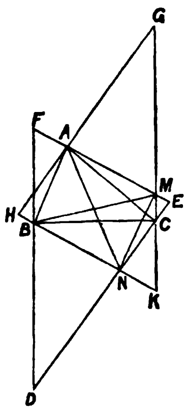
is some integer, and $\tan\phi = 0$ (measured from any radius passing through a cusp or vertex).

In (1), product of roots $= \pm(x+iy)/(x-iy) = \pm(x+iy)^2/(x^2+y^2)$;

whence, if $\beta = \tan^{-1}(y/x)$, we find $\phi = (m+n)\beta + k\pi$. Thus $\tan\phi =$ constant for all points of concurrence taken upon the symmetrical system of $m+n$ lines through O determined by $y/x = \tan\{\beta + k\pi/(m+n)\}$; and $= 0$ if the angles be measured from one of these lines.

9604. (Fannie H. Jackson.)—Two triangles are circumscribed to triangle ABC, having their sides perpendicular to the sides of a triangle ABC; prove that (1) the two triangles are equal, and find (2) their areas.

Solution by Rosa H. Whapham; Rev. J. L. Kitchin; *and others.*



1. The triangle DEF is evidently equiangular to triangle GHK; a circle will go round AMCB and also round ACNB; therefore

$$\angle AMB = \angle ACB = \angle ANB;$$

$$\therefore\ MN = \text{and} \parallel AB;\ \text{and } AM = \text{and} \parallel BN;$$

$$\therefore\ AF = NK, \text{ and } ME = HB;$$

$$\therefore\ FE = HK;\ \therefore\ \triangle DEF = \triangle GHK.$$

2. Let S be the area of one of these triangles, and $\Delta =$ triangle ABC; then

$$4S = EC\,.\,CA + FA\,.\,AB + DB\,.\,BC + HB\,.\,BA + KC\,.\,CB + GA\,.\,AC + 2\triangle ABC$$

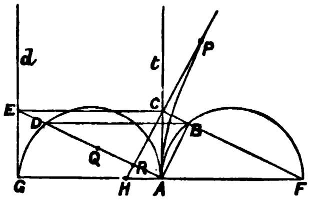
$$= b^2\,.\cot A + c^2\,.\cot B + a^2\,.\cot C + c^2\,.\cot A + a^2\,.\cot B + b^2\,.\cot C + 2\triangle ABC$$

$$= (b^2+c^2)\cot A + (c^2+a^2)\cot B + (a^2+b^2)\cot C + 2\triangle ABC;$$

$$\{a^4+b^4+c^4 - bc(b^2+c^2) - ca(c^2+a^2) - ab(a^2+b^2) + 4\Delta^2\}/2\Delta.$$

9716. (The Editor.)—If a circle pass through the vertex of a parabola and have its centre in the curve, prove that it will always touch a cissoid of Diocles, of which the generating circle is one-fourth, and the area between the curve and its asymptote three-fourths, of the circle of curvature at the vertex of the parabola.

Solution by Professor SCHOUTE; Rev. T. GALLIERS, M.A.; *and others.*

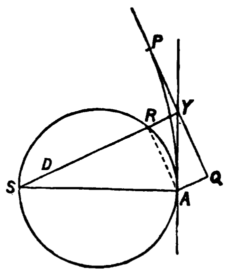


In the diagram the parabola is given by vertex A and focus F; *d* is its directrix, and *t* its tangent at the vertex. The line CH perpendicular to CF at a point C of A*t* is an arbitrary tangent; its point of contact P is found by making CP = CH.

As the join of the two points common to two circles is bisected orthogonally by the line of the centres, the two circles through A, the centres of which are P and the consecutive point of the parabola, have in common the point Q, obtained by means of the normal AR from A on CH by taking AQ = 2AR. The locus of R is the pedal of the parabola with respect to the vertex; the locus of Q is obtained by multiplying the radii vectores of this pedal for A by two.

The diagram shows that the locus of R is a cissoid; for BC = DE and BC = AR, etc. The radius of the generating circle of the cissoid described by Q is AF. The radius of the circle of curvature in A is equal to $\dfrac{2\,(\mathrm{RC})^2}{\mathrm{AR}_{\mathrm{AR}=0}} = 2\mathrm{AF}$; this proves the generating circle to be one-fourth of the circle of curvature. And then the last part of the question is a consequence of this result, the area between the curve and its asymptote being thrice the area of the generating circle (FRENET's *Exercices*, No. 432).

[Mr. E. M. LANGLEY solves the question thus:—Consider two circles whose centres are P, P′ on the parabola whose vertex is A; then the line through PP′ bisects the common chord; hence, in the limit, the ultimate point of intersection of the circle is the *image* of the vertex in the tangent to the parabola at P. The envelope is therefore a curve similar and similarly situated to the pedal of the parabola with respect to its vertex, and of twice its linear dimension.



Let the perpendicular from the focus S on the tangent at P meet the tangent at the vertex in Y, and the circle on SA as diameter in R. Draw AQ perpendicular to PY, and take SD on SY = RY.

Then AQ = RY = SD. Also AQ is parallel to SD; hence the locus of Q is congruent with that of D, and is therefore a cissoid.

For the area, draw a consecutive radius-vector SD′R′Y′; then the area between locus of D and its asymptote

$$= \mathrm{L}t\,\tfrac{1}{2}\Sigma\,(\mathrm{SY}^2-\mathrm{SD}^2)\,\angle\,\mathrm{YSY}' = \mathrm{L}t\,\tfrac{1}{2}\Sigma\,(\mathrm{SA}^2+\mathrm{AY}^2-\mathrm{RY}^2)\,\angle\,\mathrm{YSY}'$$

$$= \mathrm{L}t\,\tfrac{1}{2}\Sigma\,(\mathrm{SA}^2+\mathrm{AR}^2)\,\angle\,\mathrm{YSY}' = \tfrac{1}{4}\pi\,\mathrm{SA}^2+\mathrm{L}t\,\tfrac{1}{2}\Sigma\,\mathrm{AR}^2\,\angle\,\mathrm{RAR}'$$

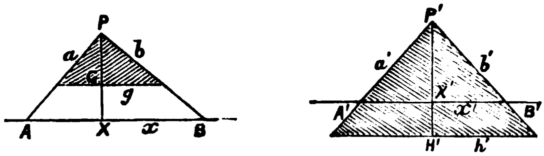
$$= \tfrac{1}{4}\pi\mathrm{SA}^2 + \text{area of circle SRA} = 3 \text{ times area of circle SRA.}$$

Area for the enveloping cissoid = 4 times that for locus of D, and area of circle of curvature at A = 16 times that of circle SRA. Hence the results.]

9751. (Professor SIRCOM, M.A.)—Show how to move a given quadrilateral into homology with another given quadrilateral, and determine the centre and axis.

Solution by Professor SCHOUTE.

In the two homographic figures (F) and (F′), of which the two given quadrilaterals *abcd* and *a′b′c′d′* are homologue quadrilaterals, we determine



the lines g and h', of which g in (F) corresponds to the line at infinity g'_∞ in (F′), and h' in (F′) to the line at infinity h_∞ in (F). Now we reckon the homographic relation determined by the correspondence of a, b, g, h_∞ and a', b', g'_∞, h' (see the diagram).

With a line x of (F) parallel to g corresponds a line x' of (F′) parallel to h'; the homographic divisions on these lines are similar, as the points at infinity correspond to one another. When for these lines x and x' the segments AB and A′B′ are equal, the superposition of (F) on (F′) with AB on A′B′ moves (F) into homology with (F′). Now, when H is the point at infinity of PG perpendicular to g, and G′, that of P′H′ perpendicular to h', we find

$$(PXGH) = (P'X'G'H'), \quad \text{or} \quad PG : XG = X'H' : P'H',$$

or

$$XG \,.\, X'H' = PG \,.\, P'H'.$$

And, as the equality of AB and A′B′ reveals the value of the quotient PX : P′X′, we find one single value of P′X′, and by it the axis x'.

The axis x' being known, we find the centre in the following manner: When the indicated superposition of (F) and (F′) is effectuated, and PP′ meets the axis x' in M, and g and h' in K and L, then the centre S on PP′ is symmetrical to M with reference to the mid-point J of KL. For the two homographic divisions on PP′ have K and L for points corresponding to infinity in the other division, and then the mid-point J of the segment KL is also the mid-point of the segment limited by the two double points.

As the permutation of the order *abcd* of the sides procures six different cases, and every case corresponds to two solutions (for A′B′ in the diagram can be replaced by the segment B″A″, wherefore B″A″B′A′ is a parallelogram with centre P′), there are in all twelve solutions.

9631. (Capitaine DE ROCQUIGNY.)—Les N premiers nombres entiers sont renfermés dans une urne; on tire au hasard deux nombres, x et y. La probabilité que la somme $x+y$ soit un nombre premier avec N est $\phi(N) : (N-1)$ ou $\phi(N) : N$ suivant que N est pair ou impair. [Suivant

l'usage, ϕ (N) désigne combien il y a de nombres inférieurs et premiers à N.]

Solution by Professor NEUBERG.

Formons une table d'addition des N premiers nombres combinés deux à deux, analogue à la table de multiplication,

$$\begin{array}{c|ccccc}
 & 1 & 2 & 3 \;\ldots\ldots & a \;\ldots\ldots & \text{N} \\
\hline
1 & 2 & 3 & 4 \;\ldots\ldots & a+1 \;\ldots\ldots & \text{N}+1 \\
2 & 3 & 4 & 5 \;\ldots\ldots & a+2 \;\ldots\ldots & \text{N}+2 \\
3 & 4 & 5 & 6 \;\ldots\ldots & a+3 \;\ldots\ldots & \text{N}+3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\text{N} & \text{N}+1 & \text{N}+2 \;\ldots\ldots\ldots\ldots & & a+\text{N} \;\ldots\ldots & 2\text{N}
\end{array}$$

Chaque nombre inscrit, à l'exception de ceux de la diagonale 2, 4, 6 ... 2N représente un tirage possible. Donc le nombre de ces tirages est N (N − 1).

Une colonne quelconque, telle que $a+1$, $a+2$ ... $a+\text{N}$, renferme ϕ (N) nombres premiers avec N ; car les restes des nombres de cette colonne, divisés par N, sont $a+1$, $a+2$ ... 0, 1 ... a. Donc la table renferme en tout Nϕ (N) nombres premiers avec N. Mais il faut effacer de la table la diagonale 2, 4, 6 ... 2N, ou les doubles des nombres 1, 2, 3 ... N. Si N est pair, les nombres de la diagonale ne sont pas premiers avec N, et le nombre des cas favorables reste égal à Nϕ (N) ; la probabilité est dès lors $\text{N}\phi(\text{N})/\text{N}(\text{N}-1) = \phi(\text{N})/\text{N}-1$. Si N est impair, la diagonale renferme ϕ (N) nombres premiers avec N ; donc parmi les tirages possibles, il n'y a plus que $\text{N}\phi(\text{N})-\phi(\text{N})$ cas favorables, et la probabilité devient $\phi(\text{N})/\text{N}$.

9356. (Professor GENESE, M.A.)—O, A_1, A_2 ... A_n are points in a straight line, and multipliers m_1, m_2 ... are chosen so that $\Sigma m_r = 0$ and $\Sigma \dfrac{m_r}{OA_r} = 0$. Prove that (1) the relation obtained is unaltered by perspective projection ; (2) if, *in addition*, $\Sigma \dfrac{m_r}{OA_r^2} = 0$, this too is unaltered by projection; (3) the theorem may be extended similarly up to $\Sigma \dfrac{m_r}{OA_r^{n-2}} = 0$; (4) in the case of $n = 3$, $-m_2 : m_3$ is the anharmonic ratio $\{\dot{O}\dot{A}_1 A_2 A_3\}$.

Solution by the PROPOSER ; Professor SIRCOM ; *and others.*

Let OV be any straight line making angle α with OA_1, $O\widehat{V}A_r = \theta_r$,

then $$\frac{OV}{OA} = \frac{\sin(\alpha+\theta_r)}{\sin\theta} = \sin\alpha\cot\theta_r + \cos\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

therefore $$OV\,\Sigma\frac{m_r}{OA_r} = \sin\alpha\,\Sigma\, m_r\cot\theta_r + \cos\alpha\,\Sigma\, m_r.$$

Using the data, we see that $\Sigma m_r \cot\theta_r = 0$. Hence $\Sigma \dfrac{m_r}{OA_r} = 0$ for all values of a and OV, or the property is projective.

Again, from (1), we have

$$OV^2 \Sigma \frac{m}{OA_r^2} = \sin^2 a\, \Sigma m_r \cot^2\theta_r + 2\sin a \cos a\, \Sigma m_r \cot\theta_r + \cos^2 a\, \Sigma m_r.$$

As before, we find $\Sigma m_r \cot^2\theta_r = 0$, if $\Sigma \dfrac{m_r}{OA_r^2} = 0$, and so on. For given positions of $OA_1A_2 \dots A_n$, it is plain that multipliers $m_1, m_2 \dots m_n$ cannot be taken to satisfy more than $n-1$ of the conditions in question.

In the case of $n = 3$, the conditions $\Sigma m_r = 0$, $\Sigma m_r/OA_r = 0$ give

$$m_2 : m_3 = \frac{1}{OA_1} - \frac{1}{OA_3} : \frac{1}{OA_2} - \frac{1}{OA_1} = \frac{A_1A_3}{OA_3} : -\frac{A_1A_2}{OA_2}$$
$$= -\{\dot{A}_1\dot{O}A_3A_2\} = -\{\dot{O}\dot{A}_1A_2A_3\}.$$

9741. (Professor NEUBERG.)—Les plans bissecteurs intérieurs des dièdres d'un tétraèdre rencontrent les arêtes opposés en six points; trouver le volume de l'octaèdre qui a pour sommets ces six points.

Solution by Professor DE WACHTER.

Each of the bisecting planes divides the edge opposite to it into two parts having the same ratio that the contiguous faces of the bisected dihedron bear to one another. If a, b, c, d be the area of the faces BCD, ACD, ABD, ABC, we arrive at the following expression:

$$\frac{\text{Octahedron}}{\text{Tetrahedron}} = 1 - \left\{ \frac{bcd}{(a+b)(a+c)(a+d)} + \frac{acd}{(b+a)(b+c)(b+d)} + \frac{abc}{(d+a)(d+b)(d+c)} + \frac{abd}{(c+a)(c+b)(c+d)} \right\}.$$

8780. (R. KNOWLES, B.A.)—From a point T (h, k) tangents TP, TQ are drawn to meet the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ in P and Q; at third tangent at L, eccentric angle θ meets these in M, N respectively; if R, R′ be the radii of the circles TPQ, TMN; prove that

$$R : R' = (a^4k^2 + b^4h^2)^{\frac{1}{2}}(ab + ak\sin\theta + bh\cos\theta) : a(1 - e^2\cos^2\theta)^{\frac{1}{2}}(a^2k^2 + b^2h^2).$$

Solution by GEORGE GOLDTHORPE STORR, M.A.

Let ϕ and ψ be the eccentric angles of P and Q respectively. The equations of the tangents at P and Q are

$$bx\cos\phi + ay\sin\phi - ab = 0, \quad bx\cos\theta + ay\sin\theta - ab = 0,$$

therefore the coordinates of M are

$$\frac{x}{a(\sin\theta-\sin\phi)} = \frac{y}{b(\cos\phi-\cos\theta)} = \frac{1}{\sin(\theta-\phi)}.$$

Hence $$MN = \frac{a\sin\frac{1}{2}(\phi-\psi)}{\cos\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}(\theta-\psi)}(1-e^2\cos^2\theta)^{\frac{1}{2}},$$

and $$PQ = 2\sin\tfrac{1}{2}(\phi-\psi)\left\{a^2\sin^2\tfrac{1}{2}(\phi+\psi)+b^2\cos^2(\phi+\psi)\right\}^{\frac{1}{2}}.$$

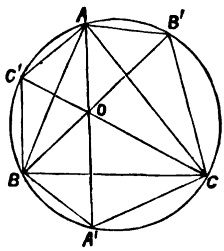
But $$\sin\tfrac{1}{2}(\phi+\psi) = \frac{ak}{(a^2k^2+b^2h^2)^{\frac{1}{2}}},\quad \cos\tfrac{1}{2}(\phi-\psi) = \frac{ab}{(a^2k^2+b^2h^2)^{\frac{1}{2}}},$$

and $$\cos\tfrac{1}{2}(\theta-\phi)\cos\tfrac{1}{2}(\phi-\psi) = \frac{bh\cos\theta+ak\sin\theta+ab}{2(a^2k^2+b^2h^2)^{\frac{1}{2}}}.$$

Hence $R : R' = PQ : MN =$ result as stated in the question.

9760. (G. G. STORR, M.A.)—If the perpendiculars from the vertices of a triangle on the opposite sides meet the circum-circle in A′, B′, C′, prove that $a.AA'+b.BB'+c.CC' = 8\Delta$.

Solution by A. M. WILLIAMS, M.A.; W. J. GREENSTREET, M.A.; *and others.*



Let the perpendiculars meet in O, then

$$\angle C'AB = \angle C'CB = \angle BAO,$$

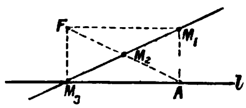
and $$\angle C'BA = \angle C'CA = \angle ABO.$$

Hence $$\Delta ABC' = \Delta ABO,$$

and $$a.AA'+b.BB'+c.CC'$$
$$= 6\Delta + 2\Delta\,(AC'B,\ AB'C,\ CA'B)$$
$$= 8\Delta.$$

9766. (Rev. T. ROACH, M.A.)—Find the locus of the centre of an ellipse touching a given line at a given point and having one focus fixed.

Solution by Professor SCHOUTE; W. J. GREENSTREET, M.A.; *and others.*



The conics touching l in A and having F for focus, have four common tangents, the two tangents through A coinciding with l and the two tangents through F, the isotropic lines through that point. Therefore the locus of their centre is a right line. Also M_1, M_2, M_3 are recognized immediately as points belonging to the locus; this line is the second diagonal M_1M_3 of the rectangle of which AF is a diagonal and l contains one side.

9587. (The EDITOR.)—**Find the locus of a point, such that, if perpendiculars be drawn from it on the sides of a triangle, the perpendiculars on two of the sides may have to one another the same ratio as the lines joining their respective feet with the foot of the perpendicular on the third side.**

Solution by Rev. J. L. KITCHIN, M.A.; *and* Prof. WOLSTENHOLME.

Let α, β, γ be the perpendiculars from P on the side of the triangle ABC; then we have

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

Now $\quad EF^2 = \beta^2+\gamma^2+2\beta\gamma\cos A,$

$\qquad FD^2 = \alpha^2+\gamma^2+2\alpha\gamma\cos B;$

hence, by the question,

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\beta^2} = \frac{\alpha^2+\gamma^2+2\alpha\gamma\cos B}{\alpha^2};$$

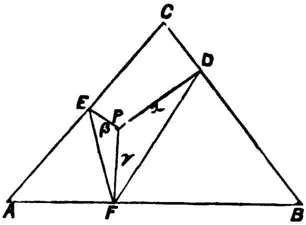


FIG. 1.

thus the equation to the locus of P is

$$(\alpha^2-\beta^2)\gamma+2\alpha\beta(\alpha\cos A-\beta\cos B) = 0,$$

or $$(2\Delta-a\alpha-b\beta)(\alpha^2-\beta^2)+2c\alpha\beta(\alpha\cos A-\beta\cos B) = 0;$$

which may now, if needful, be converted into Cartesian coordinates.

Now, when $\alpha^2 = 0$, we have either $b\beta-2\Delta = 0$ or $\beta^2 = 0$; thus the curve passes through B and C.

Also, if $\beta^2 = 0$, we have $a\alpha-2\Delta = 0$, or $\alpha^2 = 0$, which shows that the curve passes through C and A.

[The curve is a nodal circular cubic (crunode at C, the tangents bisecting C and its supplement), and its form is somewhat as in the figure 2. It passes through A, B, C and the foot of the perpendicular from C on AB, and can have only one real inflexion. Since the tangents at the node are at right angles, this cubic is the pedal of a parabola, with respect to a point on the directrix. When A = B, the cubic degenerates into a conic about ABC and the straight line bisecting C; when A = B = C, the conic is the circumcircle. This forms the most interesting particular case, and then DEF is a straight line, PE bisects the angle DPF (supposing D, E, F to be the order of the points) so that PD : PF = DE : EF. Also $\dfrac{1}{PD}+\dfrac{1}{PE}+\dfrac{1}{PF} = 0$ and $EF+FD+DE = 0$, so that $PD \cdot EF - PE \cdot FD = PF \cdot DE$, or the property is in this case true for any pair.]

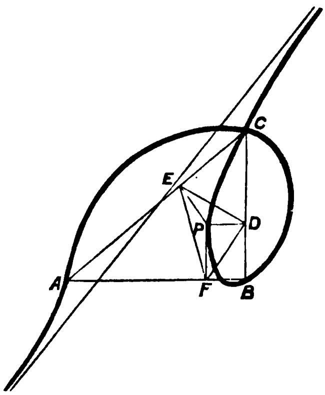


FIG. 2.

3996. (S. WATSON.)—A circle is drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle; find the chance that it does not exceed an nth part of the given circle.

Solution by D. BIDDLE.

Let $A \equiv$ the given circle, of radius $= 1$, and $B \equiv$ the circle drawn at random on A. Assuming that, in drawing B, the centre is first taken at random on the surface of A, and then the radius, of any length within the limits determined by the position of such random centre, we can divide A into two portions, namely (1) an outer ring of depth $= n^{-\frac{1}{2}}$, within which the random centre falling, the probability amounts to certainty that B will not exceed the stated dimensions, and (2) an inner circle, of radius $= 1 - n^{-\frac{1}{2}}$, as to which the probability varies. And it is easy to see that the required probability is

$$(2n^{-\frac{1}{2}} - n^{-1}) + \int_0^{1-n^{-\frac{1}{2}}} 2x\left(\frac{n^{-\frac{1}{2}}}{1-x}\right) dx = n^{-\frac{1}{2}}\left(n^{-\frac{1}{2}} - 2\log_e n^{-\frac{1}{2}}\right).$$

Thus, where $n = 4$, and $n^{-\frac{1}{2}} = \frac{1}{2}$, $P_4 = \cdot 9431472$,

where $n = 9$, and $n^{-\frac{1}{2}} = \frac{1}{3}$, $P_9 = \cdot 8435193$,

where $n = 16$, and $n^{-\frac{1}{2}} = \frac{1}{4}$, $P_{16} = \cdot 7556472$.

8063. (W. J. GREENSTREET, B.A.)—Prove that

$$\cosh 2\beta - \cos 2\alpha =$$

$$(\alpha^2 + \beta^2)\left\{1 - \frac{2(\alpha^2 - \beta^2)}{\pi^2} + \frac{(\alpha^2+\beta^2)^2}{\pi^4}\right\}\left\{1 - \frac{2(\alpha^2-\beta^2)}{2^2\pi^2} + \frac{(\alpha^2+\beta^2)^2}{2^4\pi^4}\right\} \ldots \textit{ad inf.}$$

Solution by W. N. TETLEY; ISABEL MADISON; *and others.*

$$2(\alpha^2 + \beta^2)\left\{1 - \frac{2(\alpha^2 - \beta^2)}{\pi^2} + \frac{(\alpha^2+\beta^2)^2}{\pi^4}\right\}\left\{1 - \frac{2(\alpha^2-\beta^2)}{2^2\pi^2} + \frac{(\alpha^2+\beta^2)^2}{2^4\pi^4}\right\} \ldots$$

$$= 2(\alpha + \beta i)(\alpha - \beta i)$$

$$\left\{1 - \frac{(\alpha+\beta i)^2}{\pi^2}\right\}\left\{1 - \frac{(\alpha-\beta i)^2}{\pi^2}\right\}\left\{1 - \frac{(\alpha+\beta i)^2}{2^2\pi^2}\right\}\left\{1 - \frac{(\alpha-\beta i)^2}{2^2\pi^2}\right\} \ldots$$

(taking product of alternate factors)

$$= 2\sin(\alpha + \beta i)\sin(\alpha - \beta i) = \cos 2\beta i - \cos 2\alpha = \cosh 2\beta - \cos 2\alpha.$$

9546. (Professor HUDSON, M.A.)—Evaluate $\int_{\pi/2a}^{\pi/a} \epsilon^{ax} \cos ax\, dx$, and obtain another definite integral therefrom by differentiation.

Solution by J. MacMahon, B.A.

$$\int \epsilon^{ax} \cos ax\, dx = \frac{1}{a}\epsilon^{ax}\cos ax + \int \epsilon^{ax} \sin ax\, dx$$

$$= \frac{1}{a}\epsilon^{ax}\cos ax + \frac{1}{a}\epsilon^{ax}\sin ax - \int \epsilon^{ax}\cos ax\, dx = \frac{\epsilon^{ax}}{2a}(\sin ax + \cos ax),$$

therefore definite integral $= -(\epsilon^{\pi} + \epsilon^{\frac{1}{2}\pi})/2a$.

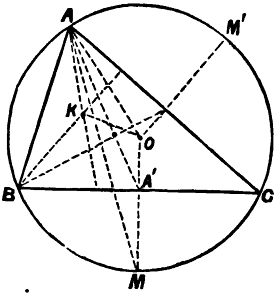
Now $$\frac{d}{da}\int_c^b \phi(x, a)\, dx = \int_c^b \frac{d}{da}\phi(xa)\,.\, dx + \phi(b)\frac{db}{da} - \phi(c)\frac{dc}{da},$$

$$\therefore\ (\epsilon^{\pi} + \epsilon^{\frac{1}{2}\pi})/2a^2 = \int_{\pi/2a}^{\pi/a} x\epsilon^{ax}(\cos ax - \sin ax)\, dx + \pi\epsilon^{\pi}/a^2; \quad \therefore \text{ \&c.}$$

9258. (Rev. T. C. Simmons, M.A.)—If θ, ϕ, ψ be the angles which the symmedians KA, KB, KC of a triangle make with the line joining K to the circumcentre, prove that $\sin\theta$, $\sin\phi$, $\sin\psi$ are respectively proportional to $\dfrac{\sin(B-C)}{\sqrt{(2b^2+2c^2-a^2)}}$, $\dfrac{\sin(C-A)}{\sqrt{(2c^2+2a^2-b^2)}}$, $\dfrac{\sin(A-B)}{\sqrt{(2a^2+2b^2-c^2)}}$,

and $\tan\theta$, $\tan\phi$, $\tan\psi$ to $\dfrac{a\sin(B-C)}{b^2+c^2-2a^2}$, $\dfrac{b\sin(C-A)}{c^2+a^2-2b^2}$, $\dfrac{c\sin(A-B)}{a^2+b^2-2c^2}$.

Solution by Professor Ignacio Beyens.



Soit A′ le milieu de BC, AM bissectrice de l'angle A.

1. Le triangle AKO nous donne

$$\sin\theta\,.\,\text{KO/A} = \sin \text{KAO}$$
$$= \sin(\text{MAA}' + \text{MAO})$$
$$= \sin \text{MAA}' \cos\tfrac{1}{2}(B-C) + \cos \text{MAA}' \sin\tfrac{1}{2}(B-C) \ldots (1),$$

Mais
$$\text{MAA}' = \text{AA}'\text{O} - \text{AMO} = 90° - \text{BA}'\text{A} - \text{AMO} = 90° - \text{BA}'\text{A} - \tfrac{1}{2}(B-C),$$

d'où
$$\sin \text{MAA}' = \cos \text{BA}'\text{A}\cos\tfrac{1}{2}(B-C) - \sin \text{BA}'\text{A}\sin\tfrac{1}{2}(B-C) \ldots\ldots (2),$$
$$\cos \text{MAA}' = \sin \text{BA}'\text{A}\cos\tfrac{1}{2}(B-C) + \cos \text{BA}'\text{A}\sin\tfrac{1}{2}(B-C) \ldots\ldots (3).$$

Mais
$$\cos \text{BA}'\text{A} = \frac{\text{AA}'^2 + \text{BA}'^2 - \text{AB}^2}{2\text{AA}'\,.\,\text{A}'\text{B}} = \frac{b^2-c^2}{a\sqrt{(2b^2+2c^2-a^2)}},$$
$$\sin \text{BA}'\text{A} = \frac{2c\sin B}{\sqrt{(2b^2+2c^2-a^2)}},$$

et posant ces valeurs dans les formules (2, 3), et après dans (1), on aura

$$\sin\theta = \frac{AO}{KO}\sin KAO = \frac{R}{KO}\cdot\frac{b^2-c^2}{a\sqrt{(2b^2+2c^2-a^2)}} = \frac{2R^2}{KO}\cdot\frac{\sin(B-C)}{\sqrt{(2b^2+2c^2-a^2)}},$$

et, d'une manière semblable, les valeurs de sin ϕ, sin ψ.

2. Le triangle AKO donne $\cos AKO = (AK^2+KO^2-R^2)/2AK.KO$; mais par la théorie des symédianes (voyez p. ex. *Über die Gegenmittellinie* von Professor H. Lieber), on a $AK = 2bct_1/(a^2+b^2+c^2)$, t_1 étant la médiane correspondante au côté BC, et par suite, comme

$$KO = R\sqrt{(1-3\tan^2\omega)},$$

$$\therefore\ \cos\theta = \frac{4b^2c^2t_1^2-3R^2\tan^2\omega\,(a^2+b^2+c^2)^2}{2AK.OK\,(a^2+b^2+c^2)^2} = \frac{b^2c^2(2b^2+2c^2-a^2)-3a^2b^2c^2}{(a^2+b^2+c^2)^2\,2AK.OK}$$

[voyez MILNE's *Companion*, p. 122, Art. 32, and p. 107, eq. (6)], et après quelques réductions très faciles on a

$$\cos\theta = \frac{bc\,(b^2+c^2-2a^2)}{KO\,(a^2+b^2+c^2)\sqrt{(2b^2+2c^2-a^2)}},$$

d'où, substituant la valeur de sin θ,

$$\tan\theta = \frac{a\sin(B-C)}{b^2+c^2-2a^2}\cdot\frac{2R^2\,(a^2+b^2+c^2)}{abc},$$

et, de la même manière, les valeurs de tan ϕ, tan ψ.

[The results may also be obtained by trilinear coordinates. For instance, the equation to KO is $abc\,(b^2-c^2)+\beta ca\,(c^2-a^2)+\gamma ab\,(a^2-b^2)=0$, and to KA, $c\beta-b\gamma=0$ (see MILNE's *Companion to Problem Papers*, p. 125, Ex. 12, &c.). The expression for sin θ has for its numerator the determinant

$$\begin{vmatrix} bc\,(b^2-c^2), & ca\,(c^2-a^2), & ab\,(a^2-b^2) \\ 0, & c, & -b \\ \sin A, & \sin B, & \sin C \end{vmatrix},$$

and for its denominator the product of two expressions of the form

$$(l^2+m^2+n^2-2mn\cos A-\ldots)^{\frac{1}{2}}.$$

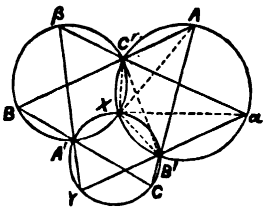
The numerator of the expression for tan θ is the same determinant, and the denominator is

$$a\,(c^4-c^2a^2-b^2a^2+b^4)+abc\,(b^2+c^2-2a^2)\cos A+bc\,(b^2-c^2)(b\cos B-c\cos C),$$

which on examination is found to be divisible by $b^2+c^2-2a^2$, and leads to the required result.]

9712. (Professor DE LONGCHAMPS.)—Un triangle ABC tourne autour d'un point fixe X de son plan. Soient A′, B′, C′ les intersections des côtés homologues de deux positions quelconques du triangle. Démontrer que le quadrilatère XA′B′C′ est toujours semblable à lui-même, et trouver comment il faut choisir le point X par rapport à ABC, pour qu'il soit le centre du cercle circonscrit ou inscrit à A′B′C′, ou l'orthocentre, ou le centre de gravité de A′B′C′.

Solution by E. M. Langley, M.A.; R. Knowles, B.A.; *and others.*



Let $\alpha\beta\gamma$ be the rotated triangle in a new position; then $\angle X\alpha\gamma = \angle X\alpha C$; therefore B′ lies on the circle through X, A, α, and $\angle X\alpha\beta = XAB$; hence C′ lies on the circle through X, A, α;

therefore $\angle XB'C' = X\alpha\beta$,

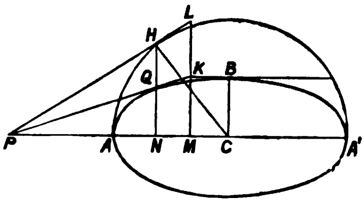
and $\angle XC'B' = XAC$;

thus the triangle XB′C′ is of constant species. Similarly for the triangles XC′A′, XA′B′.

Since XA′B′C′ is of constant species, we may for the second part of the question take its limiting position when the angle of rotation is indefinitely diminished, that is, we may take XA′, XB′, XC′ perpendicular to BC, CA, AB. Hence, according as X is the circum-centre, in-centre, ortho-centre, centroid, of A′B′C′, it is the in-centre, ortho-centre, circum-centre, symmedian point of ABC.

9770. (Maurice d'Ocagne.)—D'un point P, pris sur l'un des axes A, A′ d'une ellipse E, comme centre, on décrit un cercle C qui coupe orthogonalement le cercle bitangent à cette ellipse aux extrémités A et A′ de cet axe. Les tangentes au cercle C, perpendiculaires à AA′, forment avec les tangentes à l'ellipse E, parallèles à AA′, un rectangle dont les diagonales (qui se coupent en P) sont tangentes à l'ellipse.

Solution by G. G. Morrice, M.A., M.B.; Sarah Marks, B.Sc.; *and others.*



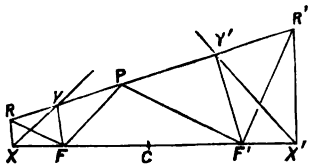
From H, one of the points of orthogonal intersection, draw the perpendicular HN on AA′, meeting the diagonal PK in Q. Let PH produced meet the side KM of the rectangle in L. The triangles LMP, CHP being equal, LM = CH = CA.

Also $QN^2 : HN^2 = KM^2 : LM^2$
$= BC^2 : CA^2$.

Therefore Q is the point of contact of the tangent from P.

9736. (Professor Wolstenholme, M.A., Sc.D.)—If Y, Y′ be the feet of the focal perpendiculars on any tangent to an ellipse, and X, X′ the feet of the corresponding directrices; prove that XY, X′Y′ will intersect on the minor axis.

Solution by J. YOUNG, M.A.; R. KNOWLES, B.A.; *and others.*



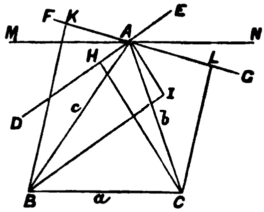
Suppose YY′ intersects the directrices in R, R′, and meets the curve at P. Let F, F′ be the foci, and C the centre. Then we have

$$\angle FXY = FRP = 90^\circ - FPR$$
$$= 90^\circ - F'PR' = F'X'Y',$$

similarly; and the lines CX, CX′ are e ual, hence, &c.

9756. (D. BIDDLE.)—A random straight line is drawn through a triangle. What chance has each side of escaping uncut?

Solution by the PROPOSER.



In order to find the chance that a particular side, say a, shall escape uncut, let a line drawn through the opposite angle, A, indicate the direction of the random line, as being parallel to it. Then the ratio of certain breadths of space will indicate the chance in the particular instance. Thus, if DE represent the direction, the chance of a escaping will be AI/HC, and if FG represent the direction, the chance will be CL/KB. If the direction, thus indicated, lie between AB and AC, there is no chance of a escaping. Draw MN parallel to a. Then the direction must lie in one of the three portions, viz., (1) between AB and AC, (2) between AB and AM, or (3) between AC and AN. In (1) the chance is *nil*; in (2) it is $c\sin(\theta - A)/b\sin\theta$, where θ varies from A to A + B; and in (3) it is $b\sin(\theta - A)/c\sin\theta$, where θ varies from A to A + C.

Such being the case, we obtain the following integral

$$P_a = \frac{1}{\pi}\left(\frac{c}{b}\int_A^{A+B}\frac{\sin(\theta - A)}{\sin\theta}\,d\theta + \frac{b}{c}\int_A^{A+C}\frac{\sin(\theta - A)}{\sin\theta}\,d\theta\right);$$

and observing that $\dfrac{c}{b} = \dfrac{\sin C}{\sin B}$, and $\dfrac{b}{c} = \dfrac{\sin B}{\sin C}$, also that

$$\sin(\theta - A) = \sin\theta\cos A - \cos\theta\sin A,$$

we find that the integral yields

$$P_a = \frac{1}{\pi}\left\{\frac{B\sin C\cos A}{\sin B} + \frac{C\sin B\cos A}{\sin C} - \frac{\sin C\sin A}{\sin B}\log\frac{\sin(A+B)}{\sin A} - \frac{\sin B\sin A}{\sin C}\log\frac{\sin(A+C)}{\sin A}\right\}.$$

P_b and P_c are represented by similar formulæ, the letters only being

changed, —in the case of P_b, A to B, B to C, C to A, and in the case of P_c, A to C, B to A, C to B.

Taking the triangle in which $A = 30°$, $B = 60°$, $C = 90°$, the above method gives

$$P_a = \tfrac{1}{3} + \tfrac{3}{8} - \frac{1}{\pi}\left(\frac{1}{\sqrt{3}}\log 2 + \frac{\sqrt{3}}{4}\log\sqrt{3}\right) = \cdot 5052372,$$

$$P_b = \tfrac{1}{8} + \tfrac{1}{6} - \frac{1}{\pi}\left(\sqrt{3}\log 2 - \frac{5\sqrt{3}}{4}\log\sqrt{3}\right) = \cdot 2880748,$$

$$P_c = 0 + 0 + \frac{1}{\pi}\left(\frac{4}{\sqrt{3}}\log 2 - \sqrt{3}\log\sqrt{3}\right) = \cdot 2066880.$$

[Mr. St. Clair has sent a solution by another method, which gives, for the same triangle, $P_a = \frac{6}{11}$, $P_b = \frac{3}{11}$, $P_c = \frac{2}{11}$.]

9282. (J. Brill, M.A.)—If $\phi(x, y)$ be a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + c^2 u = 0, \text{ then } \phi(x, y)\, e^{-\nu c^2 t}, \text{ where } \nu = \mu/\rho,$$

is the stream function of a possible two-dimensional motion of an incompressible viscous fluid.

Solution by the Proposer.

We have
$$\frac{\partial\zeta}{\partial t} + u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial y} = \nu\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right).$$

If we introduce a stream function ψ, this becomes

$$\frac{\partial\zeta}{\partial t} + \frac{\partial(\zeta, \psi)}{\partial(x, y)} = \nu\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right),$$

where ζ and ψ are connected by the equation $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} + 2\zeta = 0$. Hence, if we write $\psi = v\phi(x, y)$, where v is a function of t only, we have $\zeta = \frac{1}{2}c^2 v\phi(x, y)$. Substituting these values in the former equation, we obtain $\dfrac{dv}{dt} + \nu c^2 v = 0$, and therefore $v = e^{-\nu c^2 t}$.

9730. (Maurice d'Ocagne.)—Soit ABC un triangle inscrit dans une conique. Les côtés BC, CA, AB de ce triangle coupent un diamètre quelconque de la conique aux points A_1, B_1, C_1, dont les symétriques par rapport au centre sont A_2, B_2, C_2. Démontrer que les droites AA_2, BB_2, CC_2 concourent en un point situé sur la conique.

Solution by Professor Schoute; A. Prevost, Lic.-ès.-Sc.; *and others.*

Soient A_0, B_0, C_0 les points de la conique diamétralement opposés à A, B, C. Quand AA_2, BB_2, CC_2 passent par un même point de la conique,

A_0A_1, B_0B_1, C_0C_1 passent par le point diamétralement opposé, et récipro-

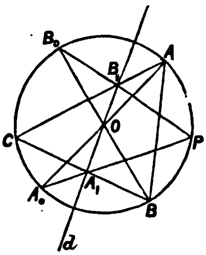


Fig. 1.

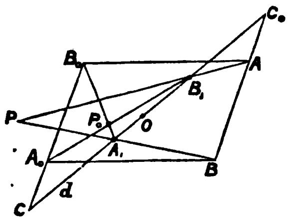


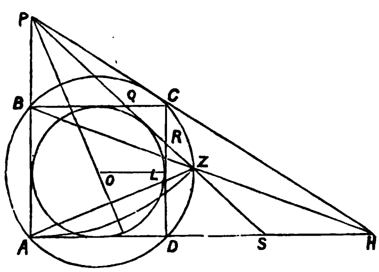
Fig. 2.

quement. Nous prouvons que A_0A_1, B_0B_1, C_0C_1 passent par un même point de la conique, et à cette fin nous n'avons qu'à démontrer que deux quelconques de ces trois droites se rencontrent sur la conique. Eh bien, le point d'intersection P (Fig. 1) de A_0A_1 et B_0B_1 est un point de la conique (ABC, A_0B_0), car les couples de côtés opposés de l'hexagone $ACBB_0PA_0$ se rencontrent en trois points situés sur le diamètre d (théorème de Pascal).

La correspondance entre les points P et P_0 (Fig. 2) est une transformation quadratique non involutive. Si C et C_0 sont les points d'intersection du diamètre d et des droites A_0B_0, AB, les points fondamentaux du système plan (P) sont A, B, C, et ceux du système plan (P_0) sont A_0, B_0, C_0. Les points de d se correspondent à eux-mêmes.

7995 & 9715. (D. Edwardes and Prof. Morel.)—Let ABCD be a square, and P a point in the production of AB through B. Let PC produced meet AD produced in H, and let S be mid-point of DH. Prove that (1) PS touches the inscribed circle of the square; and (2) PS and BH intersect on the circumcircle of the square.

Solution by the Proposers; Professor Chakravarti; *and others.*



1. Let O be the centre of the square, draw OL parallel to BC, and join OQ, OR. From the triangles HCD, CPB, we have

$$2SD : BC = BC : PB,$$

or $\quad 2SD \cdot PB = BC^2$.

But $SD : RD = BQ : BP$,

therefore $2RD \cdot BQ = BC^2$,

or $DR \cdot BQ = \frac{1}{2}BC^2 = OB^2$,

or $DR : DO = OB : BQ$,

and the sides of the triangles RDO, BQO about their equal angles are

therefore proportional; hence $\angle$ ROD = OQB = QOL; or, taking away $\angle$ ROL, we have $\angle$ QOR = LOD = $\frac{1}{2}$ a right angle; hence QR touches the inscribed circle.

2. Let Z be the point of intersection of PS and BH. Now SO bisects $\angle$ PSA; hence HB, which is parallel to SO, makes equal angles with HS and SP; therefore HS = SD = SZ, HZD is a right angle, and consequently DZB is a right angle.

9651. (S. Tebay, B.A.)—A vessel, whose content is V, is filled with wine. Water is slowly added, and supposed to thoroughly mix, the overflow being received in another vessel. Show that, if u be the quantity of water added when the two mixtures are of equal strength.

$$V = (V+u)\,e^{-u/V}.$$

Solution by the Proposer; Professor Nilkanta Sarkar; *and others.*

Let v be the quantity of wine still remaining in the vessel. When u becomes $u+du$, suppose v to become $v-dv = v-\dfrac{dv}{du}\,du.$

Now the small volume du of water added is equal to the volume of mixture discharged, which will contain $(v/V)\,du$ wine.

Therefore $$\frac{dv}{du} = -\frac{v}{V},\quad u = C - V\log v;$$

when $u=0$, $v=V$; therefore $0 = C - V\log V$; therefore

$$u = V\log(V/v),\ \text{and}\ V = e^{-u/V}.$$

Now $V-v$ = wine in second vessel, and $u(V-v)$ = water in second vessel; hence $$\frac{v}{V-v} = \frac{V-v}{u-(V-v)},\ \text{or}\ \frac{v}{V} = \frac{V-v}{u};$$

and therefore $$V = (V+u)\,e^{-u/V}.$$

9446. (J. O'Byrne Croke, M.A.)—Two radii of an ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

are at right angles, one of them being always in the plane of yz; prove that (1) a point in the other, the square of whose distance from the centre of the principal section made by that plane is equal to the difference between the squares of the radii, has for locus the sextic surface

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\left(x^2+y^2+z^2+b^2c^2\,\frac{y^2+z^2}{b^2y^2+c^2z^2}\right) = x^2+y^2+z^2;$$

and hence (2) that in this surface lie the focal loci of all sections of the ellipsoid through the axes.

Solution by the PROPOSER.

Let r be a radius vector of the ellipsoid in the plane of yz; and ρ' and ρ respectively, the radii of locus and of ellipsoid, at right angles to r and with direction cosines α, β, γ. Also, let β', γ' be the direction cosines of r, and ϕ and $\pi-\phi$ the angles made by the plane through ρr with the plane of yz; then we have

$$\cos\beta = \sin\beta'\cos\phi, \quad \cos\gamma = -\sin\gamma'\cos\phi,$$
$$\sin^2\beta' + \sin^2\gamma' = 1, \quad \cos^2\phi = \cos^2\beta + \cos^2\gamma,$$

and therefore $\cos^2\beta' = \dfrac{\cos^2\gamma}{\cos^2\beta+\cos^2\gamma}, \quad \cos^2\gamma' = \dfrac{\cos^2\beta}{\cos^2\beta+\cos^2\gamma}.$

But $r^2\left(\dfrac{\cos^2\beta'}{b^2}+\dfrac{\cos^2\gamma'}{c^2}\right) = 1, \quad \therefore\; r^2\left(\dfrac{\cos^2\gamma}{b^2}+\dfrac{\cos^2\beta}{c^2}\right) = \cos^2\beta+\cos^2\gamma;$

and $\qquad r^2 = (\cos^2\beta+\cos\gamma)\Big/\left(\dfrac{\cos^2\gamma}{b^2}+\dfrac{\cos^2\beta}{c^2}\right).$

And, since $\rho^2 = \rho'^2 + r^2$, we have

$$1\Big/\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right) = \rho'^2 + (\cos^2\beta+\cos^2\gamma)\Big/\left(\frac{\cos^2\gamma}{b^2}+\frac{\cos^2\beta}{c^2}\right);$$

therefore $1 = \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}+\left(\dfrac{\cos^2\alpha}{a^2}+\dfrac{\cos^2\beta}{b^2}+\dfrac{\cos^2\gamma}{c^2}\right)(\cos^2\beta+\cos^2\gamma)\Big/\left(\dfrac{\cos^2\gamma}{b^2}+\dfrac{\cos^2\beta}{c^2}\right).$

therefore $x^2+y^2+z^2 = \left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\left(x^2+y^2+z^2+b^2c^2\,\dfrac{y^2+z^2}{b^2y^2+c^2z^2}\right)$

is the required locus.

On this (bearing in mind the relation $\rho^2 = \rho'^2+r^2$, and the order of magnitude of a, b, c) are evidently situated the lines which are the focal loci of sections corresponding to ρ, not less than r. When ρ is not greater than r, $\sqrt{-1}$ occurring in the result obviously indicates the operation of turning round the origin of measurement in a direction at right angles to the former; so that ρ' is to be measured not along ρ, but, + and −, along b. With this correction, the equation gives the focal loci of all sections of the ellipsoid through the axes.

9575. (J. C. MALET, F.R.S.)—If the plane of a triangle ABC cut three spheres S_1, S_2, S_3 at equal angles, and if through AB a pair of tangent planes be drawn to S_3, through BC a pair to S_1, and through AC a pair to S_2; prove that the six tangent planes so drawn touch the same sphere.

Solution by the PROPOSER.

Let the coordinates of A, B, and C be respectively (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3), and let the coordinates of the centres of S_1, S_2, S_3 be respectively $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$, and their radii R_1, R_2, R_3. Now, if the six tangent planes in the question touch a sphere of which

the coordinates of the centre are (α, β, γ), and the radius R, we evidently may write

$$\alpha = \lambda_1 \alpha_1 + \mu_1 x_3 + \nu_1 x_2 = \lambda_2 \alpha_2 + \mu_2 x_1 + \nu_2 x_3 = \lambda_3 \alpha_3 + \mu_3 x_2 + \nu_3 x_1,$$
$$\beta = \lambda_1 \beta_1 + \mu_1 y_3 + \nu_1 y_2 = \lambda_2 \beta_2 + \mu_2 y_1 + \nu_2 y_3 = \lambda_3 \beta_3 + \mu_3 y_2 + \nu_3 y_1,$$
$$\gamma = \lambda_1 \gamma_1 + \mu_1 z_3 + \nu_1 z_2 = \lambda_2 \gamma_2 + \mu_2 z_1 + \nu_2 z_3 = \lambda_3 \gamma_3 + \mu_3 z_2 + \nu_3 z_1,$$
$$1 = \lambda_1 + \mu_1 + \nu_1 = \lambda_2 + \mu_2 + \nu_2 = \lambda_3 + \mu_3 + \nu_3;$$

from which we easily derive

$$\lambda_1 \Delta_1 = \lambda_2 \Delta_2 = \lambda_3 \Delta_3,$$

where
$$\Delta \equiv \begin{vmatrix} x, & x_1, & x_2, & x_3 \\ y, & y_1, & y_2, & y_3 \\ z, & z_1, & z_2, & z_3 \\ 1, & 1, & 1, & 1 \end{vmatrix} \qquad \text{.........................} \quad (1),$$

and Δ_1, Δ_2, and Δ_3 are derived from Δ by writing $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, and $(\alpha_3, \beta_3, \gamma_3)$, respectively, for (x, y, z); hence Δ_1, Δ_2, Δ_3 are proportional to P_1, P_2, P_3, where these are the perpendiculars from the centres of S_1, S_2, S_3 on the plane ABC.

Let now $\qquad x \cos\theta + y \cos\phi + z \cos\psi - p = 0$

be the equation of one of the tangents to S_1 through BC, then

$$R_1 = \alpha_1 \cos\theta + \beta_1 \cos\phi + \gamma_1 \cos\psi - p,$$

and $R = \alpha \cos\theta + \beta \cos\phi + \gamma \cos\psi - p = \lambda_1 (\alpha_1 \cos\theta + \beta_1 \cos\phi + \gamma_1 \cos\psi - p)$.

Hence $\lambda_1 = R/R_1$, and similarly $\lambda_2 = R/R_2$, $\lambda_3 = R/R_3$.

Hence, from (1), $P_1/R_1 = P_2/R_2 = P_3/R_3$; therefore, if the six tangent planes in the question touch a common sphere, the plane of the triangle ABC cuts the spheres S_1 S_2, S_3, at equal angles, and conversely.

9461. (Professor Wolstenholme, M.A., Sc.D.)—A conic S is inscribed in a given triangle ABC, its points of contact with the sides being a, b, c; another conic S′ is circumscribed to the triangle, touches S in a point O and cuts it in the points P, Q, and the two other common tangents to S, S′ intersect in R, also the tangent at O and the common chord PQ intersect in T: prove that (1) OT, OP, OQ, OR form a harmonic pencil; (2) the polars of R with respect to S, S′ concur in T and form with TO and TPQ a harmonic pencil; (3) the two common tangents from R divide OT harmonically; (4) the poles of PQ with respect to S, S′ lie upon OR and divide it harmonically; (5) the pencils A $[\dot{a}\text{BC}\dot{\text{O}}]$, A $[\dot{\text{O}}\text{BC}\dot{\text{R}}]$ are equal; as also are B $[\dot{b}\text{CA}\dot{\text{O}}]$, B $[\dot{\text{O}}\text{CA}\dot{\text{R}}]$; C $[\dot{c}\text{AB}\dot{\text{O}}]$, C $[\dot{\text{O}}\text{AB}\dot{\text{R}}]$; (6) if S be the fixed conic $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0$, and S′ variable, the straight lines OP, OQ, OR have all the same envelope, the tricusp quartic $(y+z+7x)^{-\frac{1}{2}} + (z+x+7y)^{-\frac{1}{2}} + (x+y+7z)^{-\frac{1}{2}} = 0$; (7) PQ passes through the fixed point $(x = y = z)$ in which Aa, Bb, Cc concur, hence the locus of its pole with respect to S is the straight line $x+y+z=0$; (8) the locus of the intersection of T is the nodal cubic

$$(4y+4z-5x)(4z+4x-5y)(4x+4y-5z) = 27\,(x+y+z)^3;$$

(9) the locus of R is the quartic $x^{\frac{1}{4}}+y^{\frac{1}{4}}+z^{\frac{1}{4}}=0$; *i.e.*,

$$(x^2+y^2+z^2-2yz-2zx-2xy)^2 = 128xyz\,(x+y+z),$$

and has four-point contact with the sides of the triangle at a, b, c; (10) if the tangents to S′ at A, B, C form a triangle A′B′C′; and BC, B′C′ meet in a', CA, C′A′ in b', AB, A′B′ in c', the three points a', b', c' lie on one straight line which passes through R, and is the tangent at R to the locus of R; (11) the envelope of the polar of R with respect to S is the sextic

$$(y+z-x)^{-\frac{1}{3}}+(z+x-y)^{\ \frac{1}{3}}+(x+y-z)^{-\frac{1}{3}}=0;$$

(12) the envelope of the polar of R with respect to S′ is the cubic $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$; and the loci of A′, B′, C′ are the cubics $-x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, &c. (corresponding points of the two cubics lie on a straight line through A, and their join is divided harmonically by A and BC); (13) the locus of the pole of PQ with respect to S′ is a quartic, any point of which is

given by $$\frac{x}{X\,(Y+Z+3X)}=\frac{y}{Y\,(Z+X+3Y)}=\frac{z}{Z\,(X+Y+3Z)},$$

where (XYZ) is the point O; also (14) obtain the theorems corresponding to the above, when S′ is the fixed conic $yz+zx+xy=0$, and S is variable; (1), (2), (3), (4), (5) remain the same.

Solution by Professor Swaminatha Aiyar, B.A.

Project P and Q into the circular points at infinity, and in the projection observe:—

(1) That OT is perpendicular to OR.

(2) That the polars of R are parallel to and equidistant from OT.

(3) That the portion of OT intercepted between the common tangents is bisected in O.

(4) That the line joining the centres of the touching circles is divided harmonically at O and R, the centres of similitude.

(5) Project B and C into the circular points at infinity; the conics respectively become a parabola, and a circle touching the parabola and passing through its focus. Now reciprocate with respect to the focus, and the question reduces itself to this very simple problem:—"If a circle touch a parabola at O and pass through its focus S, then the tangent at O is equally inclined to the tangent at S and the common chord."

Assume that $$l+m+n=0 \quad\text{.................................(i.).}$$

Then the line (OT), $$\frac{x}{l}+\frac{y}{m}+\frac{z}{n}=0 \quad\text{..............................(ii.),}$$

will touch the conic S (rational) = 0, at the point $\frac{x}{l^2}=\frac{y}{m^2}=\frac{z}{n^2}$. Also

the conic $$S-\left(\frac{x}{l}+\frac{y}{m}+\frac{z}{n}\right)(lx+my+nz)=0,$$

i.e., $$S'=l^3yz+m^3zx+n^3xy=0,$$

will touch S and (ii.) at that same point, and will pass through the angular points A, B, C.

(7) The common chord of S and S′ is the line

$$lx+my+nz = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii.)},$$

which passes through $x = y = z$, along with $y-z = 0$, $z-x = 0$, and $x-y = 0$; and its pole with respect to S is given by

$$\frac{x-y-z}{l} = \frac{y-z-x}{m} = \frac{z-x-y}{n}.$$

Hence the locus of the pole is $x+y+z = 0$.

(8) The locus of T is found by eliminating l, m, n from (i.), (ii.), and (iii.), and is the cubic $\quad \dfrac{x}{y-z}+\dfrac{y}{z-x}+\dfrac{z}{x-y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iv.)}.$

(6) The equation $(lm+mn+nl)\,S+(lx+my+nz)^2 = 0$,
which might be written

$$\frac{x}{l}(x-2y-2z)+\frac{y}{m}(y-2z-2x)+\frac{z}{n}(z-2x-2y) = 0,$$

represents the two tangents to S at P and Q; and, where these tangents meet (ii.), we have $\quad \dfrac{x}{y-z}+\dfrac{y}{z-x}+\dfrac{z}{x-y} = 0.$

Thus the poles of OP, OQ, OR with respect to S have the same cubic for their locus. Therefore OP, OQ, OR must have a common envelope, namely, the polar reciprocal of (iv.) with respect to S. This envelope is the curve points on which are given by

$$\frac{x}{l^2(m^2+n^2-2l^2)} = \frac{y}{m^2(n^2+l^2-2m^2)} = \frac{z}{n^2(l^2+m^2-2n^2)}.$$

Its equation can easily be put in the form given in the question.

(9) The equation of AO is $\dfrac{y}{m^2} = \dfrac{z}{n^2}$; comparing with (5) we see that AR is the line $\dfrac{y}{m^4} = \dfrac{z}{n^4}$; similarly BR is the line $\dfrac{z}{n^4} = \dfrac{x}{l^4}$. Thus R is the point $\dfrac{x}{l^4} = \dfrac{y}{m^4} = \dfrac{z}{n^4}$; its locus is therefore $x^{\frac14}+y^{\frac14}+z^{\frac14} = 0$.

(10) The tangent at R to this locus is $\dfrac{x}{l^3}+\dfrac{y}{m^3}+\dfrac{z}{n^3} = 0$, which passes through the points where $n^3y+m^3z = 0$ meets $x = 0$, &c.

(11) The pole of this tangent with respect to S is given by

$$l^3(x-y-z) = m^3(y-z-x) = n^3(z-x-y).$$

Therefore the locus of this pole, *i.e.*, the envelope of the polar of R with respect to S is $\quad (y+z-x)^{-\frac13}+(z+x-y)^{-\frac13}+(x+y-z)^{-\frac13} = 0.$

(12) The pole of the same tangent with respect to S′ is given by

$$l^3(m^3z+n^3y) = m^3(n^3x+l^3z) = n^3(m^3x+l^3y);$$

whence the pole (R′) is $\quad \dfrac{x}{l^3} = \dfrac{y}{m^3} = \dfrac{z}{n^3}.$

The envelope is thus $\quad x^{\frac13}+y^{\frac13}+z^{\frac13} = 0,$

A′ is the point $\dfrac{y}{m^3} = \dfrac{z}{n^3} = -\dfrac{x}{l^3}$; its locus is $-x^{\frac13}+y^{\frac13}+z^{\frac13} = 0$.

The lines BA′ and BR′ are $\dfrac{z}{n^3}+\dfrac{x}{l^3} = 0,\quad \dfrac{z}{n^3}-\dfrac{x}{l} = 0.$

These evidently form with BA, BC a harmonic pencil.

(13) The pole of PQ with respect to S′ is given by

$$\frac{m^3z+n^3y}{l} = \frac{n^3x+l^3y}{m} = \frac{l^3y+m^3x}{n}.$$

Hence we have $\dfrac{x}{l^2(m^2+n^2+3l^2)} = \dfrac{y}{m^2(n^2+l^2+3m^2)} = \dfrac{z}{n^2(l^2+m^2+3n^2)}.$

(14) Assume, as before, that $l+m+n = 0$; then

$$S' = (l^3x)^{\frac14}+(m^3y)^{\frac14}+(n^3z)^{\frac14} = 0,$$

$$\text{OT is } l^2x+m^2y+n^2z = 0, \quad \text{PQ is } l^4x+m^4y+n^4z = 0.$$

The point O is $\qquad lx = my = nz,$

$$\text{R is } \frac{x}{l} = \frac{y}{m} = \frac{z}{n}, \quad \text{T is } \frac{lx}{m-n} = \frac{my}{n-l} = \frac{nz}{l-m}.$$

I. The locus of R is $x+y+z = 0$; its polar with respect to S passes through the point $x = y = z$, and its polar with respect to S′ envelopes the cubic $\qquad (y+z-x)^{\frac13}+(z+x-y)^{\frac13}+(x+y-z)^{\frac13} = 0.$

II. The locus of the pole of PQ with respect to S is the quartic

$$(y+z)^{\frac14}+(z+x)^{\frac14}+(x+y)^{\frac14} = 0;$$

the locus of its pole with respect to S′ is the quartic

$$x^{-\frac12}+y^{-\frac12}+z^{-\frac12} = 0.$$

III. PQ envelopes the sextic $x^{-\frac13}+y^{-\frac13}+z^{-\frac13} = 0$; ab, bc, ca envelope the quartics $x^{-\frac12}+y^{-\frac12}+(-z)^{-\frac12} = 0$, &c.

IV. The locus of T is the nodal cubic $x(y-z)^2+y(z-x)^2+z(x-y)^2 = 0$; this cubic is also the locus of the intersections of the tangent at O and the tangents from R. OP, OQ, OR have not the same envelope; the envelope of OR is the quartic

$$(x+4y+4z)^{-\frac12}+(y+4z+4x)^{-\frac12}+(z+4x+4y)^{-\frac12} = 0.$$

V. All propositions relating to (1) anharmonic ranges and pencils, (2) collinearity of points, (3) concurrency of lines, can be easily inferred as polar reciprocals of what have already been proved in parts (1) to (13).

[The Proposer deduced these results from the particular case when S is a parabola (focus A, and B, C the circular points at infinity), and S′ a circle touching the parabola and passing through its focus, taking $Y^2 = 4a(X+a)$ for the parabola, and transforming by $X+Yi = x$, $X-Yi = y$, $4a = -z$, whence

$$\tfrac14(x-y)^2 = z\left[\tfrac12(x+y)-\tfrac14 z\right], \text{ or } x^2+y^2+z^2 = 2yz+2zx+2xy.$$

The locus of R in polar coordinates is

$$r = \tfrac14 a\sec^4\tfrac14\theta, \text{ and } x^{\frac14} = (r\cos\theta+i\,.\,r\sin\theta)^{\frac14} = r^{\frac14}(\cos\tfrac14\theta+i\sin\tfrac14\theta),$$

so $y^{\frac14} = r^{\frac14}(\cos\tfrac14\theta-i\sin\tfrac14\theta)$, whence $x^{\frac14}\pm y^{\frac14} = 2r^{\frac14}\cos\tfrac14\theta = 2(\tfrac14 a)^{\frac14} = \pm z^{\frac14}$.

It may be found, however, on trial, that nearly all are much more easily proved directly. There are still left to determine (14), the locus of the points of contact with S′ of the two tangents from R, and (15) the envelope of the tangents drawn to S′ at the points P, Q. (14) is found by eliminating $p : q : r$ from the equations

$$x/p^2+y/q^2+z/r^2=0,\quad p^3/x+q^3/y+r^3/z=0,\quad p+q+r=0;$$

and (15) is the envelope of $\dfrac{p^3x}{X^2}+\dfrac{q^3y}{Y^2}+\dfrac{r^3z}{Z^2}=0,$

subject to the relations

$$pX+qY+rZ=0,\quad p^3YZ+q^3ZX+r^3XY=0,\quad p+q+r=0.]$$

9548. (Professor Curtis, M.A.)—Prove that the following relations hold between the sines of the secondaries, from the angles to the opposite sides of a spherical triangle, the radii of the inscribed, escribed, and circumscribed circles, and the distances (spherical) of the centre of the circumcircle from those of the in- and escribed circles, calling these arcs respectively $p_1, p_2, p_3, r, r_1, r_2, r_3, \mathrm{R}, \delta, \delta_1, \delta_2, \delta_3$:

$$\frac{1}{\sin p_1}+\frac{1}{\sin p_2}+\frac{1}{\sin p_3}=\frac{\cos\delta}{\sin r\,.\cos \mathrm{R}},\quad -\frac{1}{\sin p_1}+\frac{1}{\sin p_2}+\frac{1}{\sin p_3}=\frac{\cos\delta_1}{\sin r_1\,.\cos\mathrm{R}}$$

$$\dots\quad\dots\quad\dots\quad\dots\qquad\dots\quad\dots\quad\dots\quad\dots$$

$$\frac{\cos\delta_1}{\sin r_1}+\frac{\cos\delta_2}{\sin r_2}+\frac{\cos\delta_3}{\sin r_3}=\frac{\cos\delta}{\sin r}.$$

Solution by the Proposer and Professor Promalhanath Data.

If 1, 2, 3, 4; 1′, 2′, 3′, 4′, be two sets of four points each on the surface of a sphere,

$$\begin{vmatrix} \cos(11'), & \cos(12'), & \cos(13'), & \cos(14') \\ \cos(21'), & \cos(22'), & \cos(23'), & \cos(24') \\ \dots & \dots & \dots & \dots \end{vmatrix} \equiv 0.$$

Let 1, 2, 3 be the angular points of a spherical triangle, 1′, 2′, 3′ the *poles* of the sides, 4, centre of inscribed circle, and 4′ of the circumcircle. Then

$$\cos(11')=\sin p_1,\quad \cos(12')=0,\ \&c.;$$
$$\cos(14')=\cos(24')=\cos(34')=\cos\mathrm{R};$$
$$\cos(1'4)=\cos(2'4)=\cos(3'4)=\sin r;$$
$$\cos(44')=\cos\delta.$$

Therefore
$$\begin{vmatrix} \sin p_1, & 0, & 0, & \cos\mathrm{R} \\ 0, & \sin p_2, & 0, & \cos\mathrm{R} \\ 0, & 0, & \sin p_3, & \cos\mathrm{R} \\ \sin r, & \sin r, & \sin r, & \cos\delta \end{vmatrix}=0,$$

or
$$\frac{1}{\sin p_1}+\frac{1}{\sin p_2}+\frac{1}{\sin p_3}=\frac{\cos\delta}{\sin r\cos\mathrm{R}}.$$

Similarly for the escribed circles.

9598. (J. O'Byrne Croke, M.A.)—Prove that the loci of the middle points of those parts of the generating lines of the surface

$$\frac{x^2}{a^2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{y^2+z^2}{a^2}\right)\left\{1\pm\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{\frac{1}{2}}\right\}^2,$$

lying between the axis of x and the plane of yz, are the curves traced on the sphere $4(x^2+y^2+z^2)=a^2$ by the intersecting cylinder

$$4\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1.$$

Solution by the Proposer; Professor Drox; *and others.*

The parts of the generating lines of the given surface lying between the axis of x and the plane of yz are (see Quest. 9514) each equal to a; and hence the middle points are at a distance $\frac{1}{2}a$ from the origin, and so lie upon the sphere $4(x^2+y^2+z^2)=a^2$. Again, the y and z of each middle point are, respectively, the halves of the corresponding ordinates of that extremity of the finite part a in the plane of yz and which lies on the ellipse $y^2/b^2+z^2/c^2=1$. The middle points lie also, therefore, on the cylinder $4(y^2/b^2+z^2/c^2)=1$; and, consequently, their loci are the curves traced by its intersecting surface on the sphere $4(x^2+y^2+z^2)=a^2$.

9738. (Professor Bordage.)—Construct a quadrilateral knowing the four sides and the sum of two opposite angles.

Solution by J. Young, M.A.; A. G. Cressland; *and others.*

Suppose ABCD (Fig. 1) the required quadrilateral. Take BG a fourth proportional to AD, AB, BC, and draw the perpendiculars CE, CF, GH.

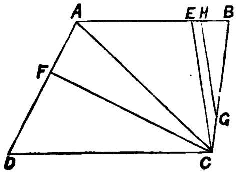


Fig. 1.

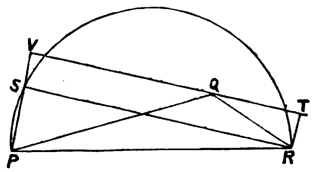


Fig. 2.

Equating the two values of the square of AC, we have

$$CD^2+AD^2-2AD\cdot DF=AB^2+BC^2-2AB\cdot BE,$$

and the rectangle AB . BE equals rectangle AD . BH. Thus, in the two right-angled triangles BGH, CDF, the hypotenuse of each is known, also the sum of the base angles B, D, and the difference of the bases DF and BH. Draw the triangle PQR (Fig. 2) having the sides PQ, QR respectively equal to CD, GB, and the contained angle $=$ B + D. On PR as diameter

describe a semicircle. Inflect PS equal to the known difference of DF, BH; join RS, draw a parallel through Q and a perpendicular through R; then PVQ, RTQ are the required triangles; and the question is completely determined.

9398. (A. E. JOLLIFFE, M A.)—When a is greater than r, find the sum to $a-r+1$ terms of $1-\frac{a^2-r^2}{(r+1)^2}+\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}}{(r+1)^2(r+2)^2}$

$$-\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}\{a^2-(r+2)^2\}}{(r+1)^2(r+2)^2(r+3)^2}+\dots$$

Solution by Professors MATZ, IGNACIO BEYENS, *and others.*

$$S=1-\frac{a^2-r^2}{(r+1)^2}+\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}}{(r+1)^2(r+2)^2}$$

$$+(-1)^{a-r}\left[\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}\dots\{a^2-(a-1)^2\}}{(r+1)^2\dots\dots a^2}\right],$$

$$k+1^{\text{th}}\text{ term}=\frac{(-1)^k}{a^2}\left[\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}\dots\{a^2-(r+k-1)^2\}}{(r+1)^2\dots(r+k-1)^2}\right.$$

$$\left.+\frac{\{a^2-r^2\}\dots\{a^2-(r+k)^2\}}{(r+1)^2\dots(r+k)^2}\right].$$

Applying this reduction to every term, we get

$$S=\frac{1}{a^2}\left[r^2+(a^2-r^2)-(a^2-r^2)-\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}}{(r+1)^2}\right.$$

$$+\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}}{(r+1)^2}+\frac{\{a^2-r^2\}\{a^2-(r+1)^2\}\{a^2-(r+2)^2\}}{(r+1)^2(r+2)^2}\dots$$

$$\left.+(-1)^{a-r}\frac{(a^2-r^2)\dots a^2-(a-1)^2}{(r+1)^2\dots(a-1)^2}+(-1)^{a-r}(0)\right]=\frac{r^2}{a^2}.$$

9528. (S. TEBAY, B.A.)—If a, b, c are conterminous edges of a tetrahedron; X, Y, Z the dihedral angles over the base;

$T^2=-4\cos S\cos(S-X)\cos(S-Y)\cos(S-Z)$, where $2S=X+Y+Z$; with similar expressions (denoted by T_1, T_2, T_3) for the other solid angles; prove that $T_1/a\sin X=T_2/b\sin Y=T_3/c\sin Z$.

Solution by the PROPOSER; Professor COCHEZ; *and others.*

Let a be the angle contained by b, c; and A_1, A_2, A_3 the areas of the faces between bc, ca, ab.

From the polar triangle, we have

$$\cos X + \cos Y \cos Z = \sin Y \sin Z \cos \alpha\,;$$

hence, squaring and reducing, we have

$$\sin^2 Y \sin^2 Z \sin^2 \alpha = 1 - \cos^2 X - \cos^2 Y - \cos^2 Z - 2 \cos X \cos Y \cos Z$$

$$= 1 - \cos^2 X - \tfrac{1}{2}(1 + \cos 2Y) - \tfrac{1}{2}(1 + \cos 2Z)$$

$$- \cos X \left\{\cos (Y+Z) + \cos (Y-Z)\right\}$$

$$= -\cos^2 X - \cos X \left\{\cos (Y+Z) + \cos (Y-Z)\right\} - \cos (Y+Z) \cos (Y-Z)$$

$$= -\left\{\cos X + \cos (Y+Z)\right\} \left\{\cos X + \cos (Y-Z)\right\}$$

$$= -4 \cos S \cos (S-X) \cos (S-Y) \cos (S-Z) = T^2\,;$$

therefore $T = \sin Y \sin Z \sin \alpha = \sin Y \sin Z / 2bcA_1 = \sin Z \sin X / 2caA_2$

$$= \sin X \sin Y / 2abA_3.$$

But it has been shown that $A_1/T_1 = A_2/T_2 = A_3/T_3$. Hence, by division,

$$T_1/a \sin X = T_2/b \sin Y = T_3/c \sin Z.$$

9417. (H. L. ORCHARD, B.Sc., M.A.)—Show, by an ordinary quadratic method, that the real roots of the equation

$$2x^{10} - 2x^8 - 20x^4 - 3x^2 + 23 = 0,$$

are $\quad +1, \; -1, \; +\left\{\sqrt{\tfrac{1}{2}} + (5\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})^{\frac{1}{2}}\right\}^{\frac{1}{2}}$, and $-\left\{\sqrt{\tfrac{1}{2}} + (5\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})^{\frac{1}{2}}\right\}^{\frac{1}{2}}$.

Solution by D. BIDDLE; B. F. FINKEL; *and others.*

Inspection of coefficients shows that two of the roots are $+1$ and -1. Again, by reduction,

$$2(x^2-1)x^8 - 20(x^2-1)x^2 - 23(x^2-1) = 0,$$

therefore $\quad x^8 - 10x^2 - \tfrac{23}{2} = 0$, and $x^8 + 2x^4 + 1 = 2x^4 + 10x^2 + \tfrac{25}{2}$,

or $\quad (x^4+1)^2 = 2(x^2 + \tfrac{5}{2})^2$, whence $x = \pm\left\{\pm\sqrt{\tfrac{1}{2}} \pm (\pm 5\sqrt{\tfrac{1}{2}} - \tfrac{1}{2})^{\frac{1}{2}}\right\}^{\frac{1}{2}}$.

Accordingly the real roots are the four stated.

9489. (ASPARAGUS.)—Prove that $\sin 2^\circ \sin 14^\circ \sin 22^\circ \sin 26^\circ \sin 34^\circ \times \sin 38^\circ \sin 46^\circ \sin 58^\circ \sin 62^\circ \sin 74^\circ \sin 82^\circ \sin 86^\circ \equiv \cdot 000244140625$.

Solution by the PROPOSER and Professor CHRISTINE FRANKLIN, M.A.

$$\sin 2^\circ \cos 2^\circ \cos 4^\circ \cos 8^\circ \cos 16^\circ \cos 32^\circ \cos 64^\circ \cos 128^\circ \cos 256^\circ \cos 512^\circ$$

$$\times \cos 1024^\circ \cos 2048^\circ \cos 4096^\circ \equiv \frac{\sin 8192^\circ}{2^{12}},$$

and $$8192^\circ = 22 \times 360^\circ + 270^\circ + 2^\circ,$$

hence $\sin 8192^\circ = -\cos 2^\circ$, and dividing out the common factor $\cos 2^\circ$, and

expressing each cosine as the sine of an acute angle, we get the identity proposed, 2^{-12} being $\equiv \cdot 000244140625$. Using 7-fig. logs., the sum of the logarithms of the sines is $\bar{4}\cdot 3876402$, which is the log. of $\cdot 0002441407$, as nearly as we can get it from 7-fig. logarithms.

9150. (R. Knowles, B.A.)—Tangents TP, TQ are drawn from a point T to meet the rectangular hyperbola $xy = a^2$ in P and Q; the circle TPQ meets the curve again in CD; if PQ be a common chord of a circle of curvature and the curve, prove that CD touches, at its mid-point, the curve $4xy = a^2$.

Solution by G. G. Storr, M.A.; Rev. T. Galliers, M.A.; *and others.*

Let P be (x', y'), the equations to PQ, CD are

$$x'^3x + a^2x'y = a^4 + x'^4, \quad a^2x + x'^2y = a^2x' \quad\ldots\ldots(1, 2);$$

hence the envelope of CD is $\quad 4xy = a^2 \ldots\ldots\ldots\ldots(3)$;

the mid-point of CD is $(x = \frac{1}{2}x', \; y = \frac{1}{2}y')$, which satisfy (3); thus CD touches at its mid-point the curve $4xy = a^2$.

9557. (H. L. Orchard, M.A., B.Sc.)—Solve, by a simple quadratic,

$$(x^2 - x)^4 + (x^2 - 2x)^4 + (x^2 - 3x + 2)^4 + 9(x+1)^4 + 7(x-2)^4 + x^8 + 16x^4 + 63 = 0.$$

Solution by R. Knowles, B.A.; Sarah Marks, B.Sc.; *and others.*

Here $\quad x^4(x-1)^4 + x^4(x-2)^4 + (x-1)^4(x-2)^4 + 9(x-1)^4$
$\qquad + 7(x-2)^4 + (x^4+7)(x^4+9) = 0,$

which $\quad \equiv \{(x-1)^4 + x^4 + 7\}\{x^4 + (x-2)^4 + 9\} = 0,$

therefore $\quad (x^2 - x + 1)^2 = -3$, and $(x^2 - 2x + 4)^2 = \frac{7}{2}$,

of which the solution can be obtained by a quadratic method.

9552. (D. Edwardes, B.A.)—Integrate the equation

$$\frac{dy}{[(y-a)(1-y^2)]^{\frac{1}{2}}} + \frac{2 \cdot 2^{\frac{1}{2}} dx}{(x^4 + 2ax^2 + 1)^{\frac{1}{2}}} = 0.$$

Solution by Professor SEBASTIAN SIRCOM, M.A.

Putting $y = 1-(1-a)\sin^2\phi$, $x = \tan\frac{1}{2}\psi$, we have

$$\frac{d\phi}{\left[1-\frac{1}{2}(1-a)\sin^2\phi\right]^{\frac{1}{2}}} = \frac{d\psi}{\left[1-\frac{1}{2}(1-a)\sin^2\psi\right]^{\frac{1}{2}}};$$

therefore $\cos\psi = \cos\phi\cos\mu + \sin\phi\sin\mu\left[1-\frac{1}{2}(1-a)\sin^2\psi\right]^{\frac{1}{2}}$,

where μ is an arbitrary constant, then

$$(1-a)^{\frac{1}{2}}(1-x^2) = (y-a)^{\frac{1}{2}}(1+x^2)\cos\mu + 2^{-\frac{1}{2}}(1-y)^{\frac{1}{2}}(x^4+2ax^2+1)^{\frac{1}{2}}\sin\mu.$$

Reversing the series as before, $x = y - y^2$, and the series gives the smaller root of this equation in y, which root is real if $x < \frac{1}{4}$. Then $y = x(1-y)^{-1}$, and, applying Lagrange's theorem,

$$(1-y)^{-w} = (1-a)^{-w} + wx(1-a)^{-w-2} + w\frac{x^2}{2!}\frac{d}{da}(1-a)^{-w-3} + \dots$$
$$+ w\frac{x^n}{n!}\left(\frac{d}{da}\right)^{n-1}(1-a)^{-w-n-1} + \dots,$$

where 0 is put for a after the differentiations, hence the result.

9638. (ASPARAGUS.)—An equilateral triangle PQR is inscribed in a given rectangular hyperbola; prove that the triangle formed by the tangents at P, Q, R will be half the triangle PQR.

Solution by R. KNOWLES, B.A.; SARAH MARKS, B.Sc.; *and others.*

Let $xy = a^2$ be the equation to the hyperbola, and x_1, y_1, &c. the coordinates of P, Q, R; and $2a^2/(y_1+y_2)$, $2y_1y_2/(y_1+y_2)$, &c. those of TT′T″ the poles of PQ, PR, QR. The

$$\begin{aligned} 2\Delta PQR &= y_1(x_2-x_3) + y_2(x_3-x_1) + y_3(x_1-x_2) \\ &= \left\{a^2y_1^2(y_2-y_3) + a^2y_2^2(y_3-y_1) + a^2y_3^2(y_1-y_2)\right\}/y_1y_2y_3 \\ &= \left\{a^2(y_1-y_2)(y_2-y_3)(y_1-y_3)\right\}/y_1y_2y_3. \end{aligned}$$

$$2\Delta TT'T'' = 4a^2(y_1-y_2)(y_2-y_3)(y_1-y_3)/(y_1+y_2)(y_2+y_3)(y_1+y_3),$$

and therefore generally,

$$\Delta PQR : \Delta TT'T'' = (y_1+y_2)(y_2+y_3)(y_1+y_3) : 4y_1y_2y_3.$$

By the question,

$$(y_1-y_2)^2 + (x_1-x_2)^2 = (y_2-y_3)^2 + (x_2-x_3)^2 = (y_1-y_3)^2 + (x_1-x_3)^2,$$

whence $\quad y_1+y_2-2y_3+a^4(y_1+y_2)/y_1^2y_2^2 - 2a^4/y_1y_2y_3 = 0,$

with two other similar relations; therefore, by addition,

$$y_3^2(y_1+y_2) + y_2^2(y_1+y_3) + y_1^2(y_2+y_3) = 6y_1y_2y_3,$$

or $\quad (y_1+y_2)(y_2+y_3)(y_1+y_3) = 8y_1y_2y_3,$

therefore $\quad \Delta PQR = 2\Delta TT'T''.$

APPENDIX I.

A STUDY IN REGARD TO THE VALUE OF Π; CONTAINING AN ACCOUNT OF SOME CURIOUS RELATIONS BETWEEN CERTAIN NUMBERS OR SERIES OF NUMBERS.

BY D. BIDDLE.

[*With an important addition furnished by* W. S. B. WOOLHOUSE, F.R.A.S.]

Starting with a well-known fact, we have

$$\frac{\pi}{4} = \tan^{-1} 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \&c.;$$

and, by coupling the terms together,

$$\frac{\pi}{4} = \frac{2}{3} + \frac{2}{35} + \frac{2}{99} + \frac{2}{195} + \frac{2}{323} + \&c.$$

$$= \frac{2}{3} + \frac{2}{3+32} + \frac{2}{3+(1+2)\,32} + \frac{2}{3+(1+2+3)\,32} + \&c.$$

Let $a = \frac{3}{32}$, $k_1 = 1$, $k_2 = 1+2 = 3$, $k_3 = 1+2+3 = 6$, &c.

Then $$4\pi = \frac{1}{a} + \frac{1}{k_1+a} + \frac{1}{k_2+a} + \frac{1}{k_3+a} + \frac{1}{k_4+a} + \&c. \quad\ldots\ldots\ldots (1).$$

This series (1) can be transformed into a single fraction of the following form, $$4\pi = \frac{A + 2Ba + 3Ca^2 + 4Da^3 + 5Ea^4 + \&c.}{Aa + Ba^2 + Ca^3 + Da^4 + Ea^5 + \&c.} \quad\ldots\ldots\ldots\ldots (2),$$

which is readily seen to be represented by

$$(A + 2Ba + \&c.) \Big/ \int_0^a (A + 2Bx + 3Cx^2 + \&c.),$$

the terms of the numerator being the differential coefficients of the respective terms of the denominator, at the upper limit; and in which A, B, C, D, &c. vary with the number of terms utilised in (1), but always bear a certain relation to each other. Thus, if $n+1$ = the number of terms taken,

$$A_n = k_n(A_{n-1}), \quad B_n = k_n(B_{n-1}) + A_{n-1},$$

$$C_n = k_n(C_{n-1}) + B_{n-1}, \quad D_n = k_n(D_{n-1}) + C_{n-1},$$

and so on; and we are able to write down the successive values as follows:—

n	A	B	C	D	E	F
1.	1	1				
2.	3	4	1			
3.	18	27	10	1		
4.	180	288	127	20	1	
5.	2700	4500	2193	427	35	1
6.	56700	97200	50553	11160	1162	56
	&c.		&c.	&c.		

.... (3).

From this it further appears that

$$\left.\begin{aligned} A_n &= k_1 . k_2 . k_3 \ldots k_n = p_n \\ B_n &= p_n\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots \frac{1}{k_n}\right) = p_n q_n \\ C_n &= p_n\left(0 + \frac{q_1}{k_2} + \frac{q_2}{k_3} + \ldots \frac{q_{n-1}}{k_n}\right) = p_n r_n \\ D_n &= p_n\left(0 + 0 + \frac{r_1}{k_3} + \frac{r_2}{k_4} + \ldots \frac{r_{n-2}}{k_n}\right) = p_n s_n \end{aligned}\right\} \text{......... (4),}$$

and so on. In each case the significant terms alone are counted.

Now $\qquad k_n = \frac{1}{2}n(n+1)$;

therefore $\quad p_n = \frac{1}{2}(n+1)n \,.\, \frac{1}{2}n(n-1) \,.\, \frac{1}{2}(n-1)(n-2) \ldots 6 \,.\, 3 \,.\, 1$

$$= (\tfrac{1}{2})^n (n+1\,(n!)^2 = A_n \text{ .... (5).}$$

Moreover, since $\quad q_n = \dfrac{1}{k_1} + \dfrac{1}{k_2} + \ldots \dfrac{1}{k_n}$,

it is easy to see that $\quad q_n = \dfrac{n^2}{k_n} = \dfrac{2n}{n+1} = B : A$,

from which it follows that, when n is infinite, $B = 2A$ .... (6).

But, since $\quad q_n = \dfrac{2n}{n+1} = 1 + \dfrac{n-1}{n+1}$,

we can form a useful series from which to derive r_n; for

$$q_n = 1 + \left(\frac{1}{3},\ \frac{2}{4},\ \frac{3}{5},\ \frac{4}{6},\ \frac{5}{7},\ \text{\&c.}\right) \text{............... (7),}$$

according as $n = 2, 3, 4$, &c., the fraction within brackets rising to unity as n becomes infinite.

Now $\quad r_n = \dfrac{q_1}{k_2} + \dfrac{q_2}{k_3} + \dfrac{q_3}{k_4} + \text{\&c.} = \dfrac{1}{3} + \dfrac{1+\frac{1}{3}}{6} + \dfrac{1+\frac{2}{4}}{10} + \text{\&c.}$

But $\quad q = 1 + \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{10} + \dfrac{1}{15} + \text{\&c.} = 2$;

therefore $\quad \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{10} + \dfrac{1}{15} + \text{\&c.} = 1$ .... (8),

and $\quad r_\infty = 1 + \dfrac{1}{3 \,.\, 6} + \dfrac{2}{4 \,.\, 10} + \dfrac{3}{5 \,.\, 15} + \text{\&c.}$ .... (9),

$\qquad = $ approximately $1{\cdot}42$.

But, as a further aid to the investigation, it is important to observe that at whatever stage we take A, B, C, D, &c., their differences vanish by continuous subtraction. Thus, when $A = 18$, $B = 27$, $C = 10$, $D = 1$, we at once see that $A - \{B - (C - D)\} = 0$, and in reverse fashion also $D - \{C - (B - A)\} = 0$. This holds good whenever, in (3), we continue the subtraction through all the coefficients; and it shows that not only $B > A$, but $C > B - A$, $D > C - (B - A)$, &c. .... (10).

Moreover, there is a relation between the several differences and their increments in the several stages. Taking A as unity, $B - A$ in the various stages is $0, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}$, &c., the respective increments being $\frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}$, &c. We can therefore represent the difference as follows:—

$$B_\infty - A_\infty = \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_4} + \frac{1}{k_5} + \&c. = 1 \quad \text{............ (11).}$$

Now, $C - (B - A)$ in the various stages is $0, \frac{1}{18}, \frac{19}{180}, \frac{393}{2700}$, &c., the respective increments being $\frac{1}{18}, \frac{1}{20}, \frac{1}{25}, \frac{2}{63}$, &c., which can be represented by

$$\frac{1}{3\,.\,6} + \frac{3}{6\,.\,10} + \frac{6}{10\,.\,15} + \frac{10}{15\,.\,21} + \&c.,$$

$$= \frac{k_1}{k_2 k_3} + \frac{k_2}{k_3 k_4} + \frac{k_3}{k_4 k_5} + \&c. \quad \text{........................ (12).}$$

But $C - (B - A)$ as compared with $B - A$ has for its increment at each stage $(B - A)_n \,.\, 1/k_{n+1}$. Similarly, $D - \{C - (B - A)\}$ has for its increment $\{C - (B - A)\}_n \,.\, 1/k_{n-1}$. Consequently, referring the values at each stage to A of the same stage as unity, we have

$$(1/k_n)\,(B - A)_{n-1} = \{C - (B - A)\}_n - \{C - (B - A)\}_{n-1},$$

$$(1/k_n)\,\{C - (B - A)\}_{n-1} = [D - \{C - (B - A)\}]_n - [D - \{C - (B - A)\}]_{n-1}, \ \&c.$$

And to this we can bring the fact, already found, that

$$(1/k_n)\,A_{n-1} = B_n - B_{n-1}, \quad (1/k_n)\,B_{n-1} = C_n - C_{n-1}, \ \&c.;$$

whence we further obtain

$$A_n = k_n B_n - k_n^2 C_n + k_n^3 D_n - k_n^4 E_n + \&c. \quad \text{........................ (13),}$$

and $$A_{n-1} = k_{n-1} B_{n-1} - k_{n-1}^2 C_{n-1} + k_{n-1}^3 D_{n-1} - k_{n-1}^4 E_{n-1} + \&c. \quad \text{.... (14),}$$

whence $$A_n = k_n k_{n-1} B_{n-1} - k_n k_{n-1}^2 C_{n-1} + k_n k_{n-1}^3 D_{n-1} - k_n k_{n-1}^4 E_{n-1} + \&c.$$

$$= k_{n-1}(B_n - A_{n-1}) - k_{n-1}^2 (C_n - B_{n-1}) + k_{n-1}^3 (D_n - C_{n-1}) - \&c.$$

But, by (14), $A_{n-1} - k_{n-1} B_{n-1} + k_{n-1}^2 C_{n-1} - \&c. = 0$,

therefore $$A_n = k_{n-1} B_n - k_{n-1}^2 C_n + k_{n-1}^3 D_n - k_{n-1}^4 E_n + \&c. \quad \text{............ (15).}$$

And by a similar process of reasoning, we have

$$A_n = B_n - C_n + D_n - E_n + F_n - \&c. \quad \text{.................. (16),}$$

in which k is reduced step by step to unity, from (13) through (15) to (16). Any *lower* grade of k than k_n will serve equally, but no higher grade (such as k_{n+1}), nor any number which is not a grade of k (such as 2, 4, 5, &c.) Where n is infinite, all grades of k are admissible.

$$\left.\begin{aligned}
A &= B - C + D - E + F - G + \&c.\\
A &= 3B - 3^2C + 3^3D - 3^4E + 3^5F - 3^6G + \&c.\\
A &= 6B - 6^2C + 6^3D - 6^4E + 6^5F - 6^6G + \&c.\\
A &= 10B - 10^2C + \&c.
\end{aligned}\right\} \dots\dots\dots\dots (17).$$

And by manipulations of these, we have

$$\left.\begin{aligned}
A &= B - C + D - E + F - G + \&c.\\
B &= 4C - 13D + 40E - 121F + 364G - \&c.\\
C &= 10D - 73E + 478F - 2989G + \&c.\\
D &= 20E - 273F + 3208G - \&c.
\end{aligned}\right\} \dots\dots\dots\dots (18),$$

which can be written down without trouble, by observing a certain law, viz., that each succeeding line is formed by adding the coefficients immediately above to the product of k_r, and the coefficient of the preceding term in the line itself that is being formed.

Thus

$$\left.\begin{aligned}
B &= (3+1)\,C - \{3\,(3+1)+1\}\,D + \&c.\\
C &= (6+4)\,D - \{6\,(6+4)+13\}\,E + \&c.\\
D &= (10+10)\,E - \{10\,(10+10)+73\}\,F + \&c.
\end{aligned}\right\} \dots\dots\dots (19).$$

By substituting the values thus obtained, we can eliminate the earlier terms. Thus we can represent the values of A and B in terms beginning with C:

$$\left.\begin{aligned}
A &= 3C - 12D + 39E - 120F + 363G - \&c.\\
B &= 4C - 13D + 40E - 121F + 364G - \&c.
\end{aligned}\right\} \dots\dots\dots\dots (20).$$

Also, A, B, C, in terms beginning with D:

$$\left.\begin{aligned}
A &= 18D - 180E + 1314F - 8604G + \&c.\\
B &= 27D - 252E + 1791F - 11592G + \&c.\\
C &= 10D - 73E + 478F - 2989G + \&c.
\end{aligned}\right\} \dots\dots\dots\dots (21),$$

and so on.

Now we have already found that $B_\infty = 2A_\infty$.

Consequently, $C - D + E - F + G - H + \&c. = A$;

and $3C - 3D + 3E - 3F + 3G - 3H + \&c. = 3A$.

But $3C - 12D + 39E - 120F + 363G - 1092H + \&c. = A$,

therefore $D = \frac{2}{9}A + 4E - 13F + 40G - 121H + \&c. \dots\dots\dots\dots (22)$,

and $C = (1 + \frac{2}{9})\,A + 3E - 12F + 39G - 120H + \&c. \dots\dots\dots\dots (23)$.

It is a mere matter of labour to convert all the terms thus into corresponding values of A.

Another important relation between the values is found as follows:—

$$\left.\begin{aligned}
B &= 4C - 13D + 40E - 121F + 364G - \&c.\\
\tfrac{4}{3}A &= 4C - 16D + 52E - 160F + 484G - \&c.\\
\text{therefore}\quad B - \tfrac{4}{3}A &= 3D - 12E + 39F - 120G + \&c.\\
\text{whilst}\quad A &= 3C - 12D + 39E - 120F + \&c.
\end{aligned}\right\} \dots\dots\dots\dots (24).$$

$$\left.\begin{aligned}
\text{Again,}\quad B &= 27D - 252E + 1791F - 11592G + \&c.\\
\tfrac{3}{2}A &= 27D - 270E + 1971F - 12906G + \&c.\\
\text{therefore}\quad B - \tfrac{3}{2}A &= 18E - 180F + 1314G - \&c.\\
\text{whilst}\quad A &= 18D - 180E + 1314F - \&c.
\end{aligned}\right\} \dots\dots\dots\dots (25).$$

$$
\left.\begin{array}{lrl}
\text{Once more,} & B &= 288E-5580F+75024G-874764H+\&c. \\
 & \tfrac{8}{5}A &= 288E-5760F+78624G-923904H+\&c. \\
\text{therefore} & B-\tfrac{8}{5}A &= 180F-3600G+49140H-\&c. \\
\text{whilst} & A &= 180E-3600F+49140G-\&c.
\end{array}\right\}\ldots\ldots (26).
$$

Moreover, by addition of the two last lines in (24), we have

$$
\left.\begin{array}{lrl}
 & \tfrac{2}{3}A &= 3C-9D+27E-81F+243G-\&c. \\
\text{whilst} & A &= 3B-9C+27D-81E+243F-\&c.
\end{array}\right\}\ldots\ldots\ldots\ldots\ldots (27).
$$

This maintenance of the same coefficients, with a shifting (one step forward) of letters, is very interesting, and ought to be of assistance in determining the values of the letters. We have a further instance in the following :—

$$
\left.\begin{array}{rl}
A &= 3C-12D+39E-120F+363G-\&c. \\
\tfrac{2}{3}A &= 3D-12E+39F-120G+363H-\&c. \\
C-\tfrac{11}{9}A &= 3E-12F+39G-120H+363I-\&c.
\end{array}\right\}\ \ldots\ldots\ldots (28).
$$

But, taking (24), (25), (26), we observe a gradation in the commencing coefficients, viz., 3, 18, 180, which are p_2, p_3, p_4 [*vide* (4)]. We also observe a gradation in the values on the other side of the equation; for

$$
B-\tfrac{4}{3}A = \tfrac{2}{3}A, \quad B-\tfrac{3}{2}A = \tfrac{1}{2}A, \quad B-\tfrac{8}{5}A = \tfrac{2}{5}A,
$$

and $\tfrac{2}{3}$, $\tfrac{1}{2}$, $\tfrac{2}{5}$ belong to the series $2/(n+2)$. Besides which, the second term on the right has for its coefficient a number which is the product of the first and a factor formed by addition of grades of k. Thus, in (24), $12 = 3(k_1+k_2)$; in (25), $180 = 18(k_1+k_2+k_3)$; in (26), $3600 = 180(k_1+k_2+k_3+k_4)$. Now this factor is easily found for any value of n; for

$$
k_1+k_2+\ldots+k_n = \Sigma_1^n \frac{n(n+1)}{2} = \Sigma\frac{n^2}{2} + \Sigma\frac{n}{2} = \frac{n(n+1)(n+2)}{6},
$$

as found in the several first terms on the right in (18). The second terms in (18) afford a similar factor for the *third* terms in the several series now under consideration; the third, a similar factor for the present *fourth*; and so on. It is therefore a mere matter of labour to produce any number of these duplicate series.

On referring to (4), (11), and (12), we find that

$$
B_\infty = 2A_\infty, \quad C_\infty = B_\infty - A_\infty + \frac{1}{3.6} + \frac{1}{3.10} + \frac{1}{6.10} + \&c.,
$$

$$
D_\infty = C_\infty - (B_\infty - A_\infty) + \frac{1}{3.6.10} + \&c.,
$$

$$
E_\infty = D_\infty - \{C_\infty - (B_\infty - A_\infty)\} + \frac{1}{3.6.10.15} + \&c.,
$$

and so on. Let

$$
m_1 = \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \&c., \quad m_2 = \frac{1}{3.6} + \frac{1}{3.10} + \frac{1}{6.10} + \&c.,
$$

$$
m_3 = \frac{1}{3.6.10} + \frac{1}{3.6.15} + \frac{1}{3.10.15} + \frac{1}{6.10.15} + \&c.,
$$

$$
m_4 = \frac{1}{3.6.10.15} + \frac{1}{3.6.10.21} + \&c., \&c.
$$

$$\left.\begin{array}{lll} \text{Then} \quad A_\infty = 0+1, & B_\infty = 1+m_1, & C_\infty = m_1+m_2 \\ D_\infty = m_2+m_3, & E_\infty = m_3+m_4, & F_\infty = m_4+m_5 \\ \&c. & \&c. & \&c. \end{array}\right\} \ldots\ldots (29).$$

We can now utilise the arrangements exemplified in (24)—(28), observing that $B_\infty - C_\infty = (1-m_2)\,A_\infty$, $C_\infty - D_\infty = (m_1 - m_3)\,A_\infty$,

$D_\infty - E_\infty = (m_2 - m_4)\,A_\infty$, &c.

Thus, if we take (27) to begin with, and subtract one equation from the other, we have

$$3\,(B-C) - 9\,(C-D) + 27\,(D-E) - 81\,(E-F) + 243\,(F-G) - \&c. = -\tfrac{2}{3},$$

or

$$3(1-m_2) - 9(m_1-m_3) + 27(m_2-m_4) - 81(m_3-m_5) + 243(m_4-m_6) - \&c. = -\tfrac{2}{3},$$

whence $\quad m_2 - 3m_3 + 9m_4 - 27m_5 + 81m_6 - \&c. = \tfrac{2}{9}$ .... (30).

By applying (29) to (17), we also have

$$m_2 - 6m_3 + 36m_4 - 216m_5 + 1296m_6 - \&c. = \tfrac{5}{36} \;\ldots\ldots\ldots\; (31),$$

$$m_2 - 10m_3 + 100m_4 - 1000m_5 + 10000m_6 - \&c. = \tfrac{9}{100} \;\ldots\; (32),$$

and so on, the value of each such series being expressed by the fraction

$$(k_x - 1)/k_x^2.$$

By subtracting (31) from (30) and dividing, we have

$$m_3 - 9m_4 + 63m_5 - 405m_6 + \&c. = \tfrac{1}{36} \;\ldots\ldots\ldots\ldots\ldots\ldots\; (33),$$

and we can eliminate any number (n) of terms in the same way, by taking ($n+1$) series of the type represented by (30)—(32).

Thus, by subtracting (32) from (30) and dividing, we have

$$m_3 - 13m_4 + 139m_5 - 1417m_6 + \&c. = \tfrac{17}{900} \;\ldots\ldots\ldots\ldots\ldots\; (34).$$

Then, by subtracting (34) from (33), we obtain

$$m_4 - 19m_5 + 253m_6 - 2935m_7 + \&c. = \tfrac{1}{450} \;\ldots\ldots\ldots\ldots\ldots\; (35).$$

But we can take (30), (31), (32), and transform them as follows :—

$$m_2 - 3x = \tfrac{2}{9}, \quad m_2 - 6y = \tfrac{5}{36}, \quad m_2 - 10z = \tfrac{9}{100} \ldots\ldots\ldots (36-38),$$

in which

$$x = m_3 - 3m_4 + 9m_5 - 27m_6 + \&c.,$$
$$y = m_3 - 6m_4 + 36m_5 - 216m_6 + \&c.,$$
$$z = m_3 - 10m_4 + 100m_5 - 1000m_6 + \&c.$$

And putting down the gradations through which x, y, z have already passed, when m_1, m_2 began the respective series, we have

$$\text{In place of } x \left\{ \begin{array}{l} \tfrac{1}{3} = m_1 - 3m_2 + 9m_3 - 27m_4 + \&c. \\ \tfrac{2}{9} = m_2 - 3m_3 + 9m_4 - 27m_5 + \&c. \end{array} \right.$$

$$\text{In place of } y \left\{ \begin{array}{l} \tfrac{1}{6} = m_1 - 6m_2 + 36m_3 - 216m_4 + \&c. \\ \tfrac{5}{36} = m_2 - 6m_3 + 36m_4 - 216m_5 + \&c. \end{array} \right.$$

$$\text{In place of } z \left\{ \begin{array}{l} \tfrac{1}{10} = m_1 - 10m_2 + 100m_3 - 1000m_4 + \&c. \\ \tfrac{9}{100} = m_2 - 10m_3 + 100m_4 - 1000m_5 + \&c. \end{array} \right.$$

It will be interesting to find the law governing the succeeding values.

Now, m_2, m_3, &c. are compound series of a kind difficult to sum. For convenience, m_2 may be rendered as follows :—

$$\begin{aligned}
&\frac{1}{3.6}+\frac{1}{3.10}+\frac{1}{3.15}+\frac{1}{3.21}+\frac{1}{3.28}+\frac{1}{3.36}+\&c.\\
&\qquad+\frac{1}{6.10}+\frac{1}{6.15}+\frac{1}{6.21}+\frac{1}{6.28}+\frac{1}{6.36}+\&c.\\
&\qquad\qquad+\frac{1}{10.15}+\frac{1}{10.21}+\frac{1}{10.28}+\frac{1}{10.36}+\&c.\\
&\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{aligned}$$

And since $$\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\frac{1}{21}+\&c.=1,$$

we can sum the terms of m_2 horizontally, thus—

$$\frac{1}{3}\cdot\frac{2}{3}+\frac{1}{6}\cdot\frac{1}{2}+\frac{1}{10}\cdot\frac{2}{5}+\frac{1}{15}\cdot\frac{1}{3}+\&c. \;\ldots\ldots\ldots\; (39),$$

and we can sum them vertically as follows—

$$\frac{1}{3}\cdot\frac{1}{6}+\frac{1}{2}\cdot\frac{1}{10}+\frac{3}{5}\cdot\frac{1}{15}+\frac{2}{3}\cdot\frac{1}{21}+\frac{5}{7}\cdot\frac{1}{28}+\&c.\ldots\ldots (40).$$

These sums are necessarily equal. Adding them together, term by term, we have

$$2m_2=\left(\frac{1}{6}+\frac{2}{3.6}\right)+\left(\frac{1}{10}+\frac{2}{6.10}\right)+\left(\frac{1}{15}+\frac{2}{10.15}\right)+\&c.\ldots(41),$$

therefore

$$\left.\begin{aligned}
m_2&=\frac{1}{2}\left(\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots\right)+\left(\frac{1}{3.6}+\frac{1}{6.10}+\frac{1}{10.15}+\ldots\right)\\
&=\frac{1}{1.3}+\frac{1}{3.6}+\frac{1}{6.10}+\frac{1}{10.15}+\frac{1}{15.21}+\&c.
\end{aligned}\right\}\ldots (42).$$

We therefore know by observation of m_2, as rendered at head of page, that

$$\left(\frac{1}{3.10}+\frac{1}{3.15}+\ldots\right)+\left(\frac{1}{6.15}+\frac{1}{6.21}+\ldots\right)+\&c.=\frac{1}{3}\ldots (43).$$

On completing the square of which m_2 forms part, we have

$$\left.\begin{aligned}
&\frac{1}{3.3}+\frac{1}{3.6}+\frac{1}{3.10}+\frac{1}{3.15}+\frac{1}{3.21}+\&c.\\
&\frac{1}{6.3}+\frac{1}{6.6}+\frac{1}{6.10}+\frac{1}{6.15}+\frac{1}{6.21}+\&c.\\
&\frac{1}{10.3}+\frac{1}{10.6}+\frac{1}{10.10}+\frac{1}{10.15}+\frac{1}{10.21}+\&c.\\
&\frac{1}{15.3}+\frac{1}{15.6}+\frac{1}{15.10}+\frac{1}{15.15}+\frac{1}{15.21}+\&c.\\
&\frac{1}{21.3}+\frac{1}{21.6}+\frac{1}{21.10}+\frac{1}{21.15}+\frac{1}{21.21}+\&c.\\
&\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{aligned}\right\}\ldots\ldots\ldots (44).$$

And we at once see that it is symmetrical on the two sides of the diagonal $\frac{1}{3 \, . \, 3}$, $\frac{1}{6 \, . \, 6}$, &c. It is, in fact, the product of

$$\left(\frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \&\text{c.}\right)$$

multiplied by itself, and equals 1. Consequently

$$m_2 = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{15^2} + \&\text{c.}\right) \text{.........} \quad (45).$$

By (42), $$m_2 = \Sigma_1^\infty \frac{4}{n\,(n+1)^2(n+2)} \text{..............................} \quad (46).$$

By (45), $$m_2 = \frac{1}{2} - \frac{1}{2}\,\Sigma_1^\infty \frac{4}{(n+1)^2(n+2)^2} \text{..........................} \quad (47).$$

The summation of these series is not to be found in the ordinary text-books, at least in any form that does not presuppose π; but (42) and (45) enable us, by a series of easy calculations, to find in succession that m_2 lies between $\frac{1}{3}$ and $\frac{1}{2}$, $\frac{7}{18}$ and $\frac{8}{18}$, &c.; also that it lies between $\frac{3}{9}$ and $\frac{4}{9}$, $\frac{28}{72}$ and $\frac{31}{72}$, &c. In the former we proceed by the addition and subtraction respectively of $1/k_n k_{n+1}$ and $1/2k_n^2$; in the latter, of $1/k_n k_{n-1}$ and $1/2k_{n+1}^2$. Consequently the correct value of m_2 is *more* than a third of their difference from the larger of the two fractions in the former case, but in the latter case *less* than a third. Now, after about a score of additions and subtractions, the limiting results by the

former method are ·4201091 and ·4203470 .... (48*a*),

after which ·4201297 and ·4203357 .... (48*b*);

whilst by the latter method the limiting results are

·4201091 and ·4203357 .... (49*a*),

after which ·4201297 and ·4203263 .... (49*b*).

Taking a third of the difference in (48*a*), we have ·4202677, and in (49*a*) ·4202602; whilst in (48*b*) we have ·4202670, and in (49*b*) ·4202608. Midway between the results in either case, is ·4202639, which we may regard as slightly in excess of the correct value. At any rate

$$m_2 = \cdot 420263\ldots \qquad \text{............} \quad (50).$$

And, by (45), $$\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \&\text{c.}$$

$$= 4\left(\frac{1}{2^2 \, . \, 3^2} + \frac{1}{3^2 \, . \, 4^2} + \frac{1}{4^2 \, . \, 5^2} + \&\text{c.}\right) = 15947\ldots \qquad \text{......} \quad (51).$$

In considering m_3, we may write it as follows :—

$$\left.\begin{array}{l}\frac{1}{3.6.10}+\frac{1}{3.6.15}+\frac{1}{3.6.21}+\frac{1}{3.6.28}+\frac{1}{3.6.36}+\&c.\\ \qquad+\frac{1}{3.10.15}+\frac{1}{3.10.21}+\frac{1}{3.10.28}+\frac{1}{3.10.36}+\&c.\\ \qquad\qquad+\frac{1}{3.15.21}+\frac{1}{3.15.28}+\frac{1}{3.15.36}+\&c.\\ \qquad\qquad\qquad+\frac{1}{3.21.28}+\frac{1}{3.21.36}+\&c.\\ \qquad\qquad\qquad\qquad+\frac{1}{3.28.36}+\&c.\end{array}\right\}(52)$$

$$\left.\begin{array}{l}+\frac{1}{6.10.15}+\frac{1}{6.10.21}+\frac{1}{6.10.28}+\frac{1}{6.10.36}+\&c.\\ \qquad+\frac{1}{6.15.21}+\frac{1}{6.15.28}+\frac{1}{6.15.36}+\&c.\\ \qquad\qquad+\frac{1}{6.21.28}+\frac{1}{6.21.36}+\&c.\\ \qquad\qquad\qquad+\frac{1}{6.28.36}+\&c.\end{array}\right\}(53)$$

$$\left.\begin{array}{l}+\frac{1}{10.15.21}+\frac{1}{10.15.28}+\frac{1}{10.15.36}+\&c.\\ \qquad+\frac{1}{10.21.28}+\frac{1}{10.21.36}+\&c.\\ \qquad\qquad+\frac{1}{10.28.36}+\&c.\end{array}\right\}(54)$$

$$\left.\begin{array}{l}+\frac{1}{15.21.28}+\frac{1}{15.21.36}+\&c.\\ \qquad+\frac{1}{15.28.36}+\&c.\end{array}\right\}(55)$$

$$+\frac{1}{21.28.36}+\&c.\Big\}(56),$$

which may be continued indefinitely.

Adding the terms horizontally, we have

$$\frac{1}{18}\cdot\frac{1}{2}+\frac{1}{30}\cdot\frac{2}{5}+\frac{1}{45}\cdot\frac{1}{3}+\frac{1}{63}\cdot\frac{2}{7}+\frac{1}{84}\cdot\frac{1}{4}+\&c.\ \ldots\ (52a),$$

$$+\frac{1}{60}\cdot\frac{2}{5}+\frac{1}{90}\cdot\frac{1}{3}+\frac{1}{126}\cdot\frac{2}{7}+\frac{1}{168}\cdot\frac{1}{4}+\&c.\ \ldots\ (53a),$$

$$+\frac{1}{150}\cdot\frac{1}{3}+\frac{1}{210}\cdot\frac{2}{7}+\frac{1}{280}\cdot\frac{1}{4}+\&c.\ \ldots\ (54a),$$

$$+\frac{1}{315}\cdot\frac{2}{7}+\frac{1}{420}\cdot\frac{1}{4}+\&c.\ \ldots\ (55a),$$

$$+\frac{1}{588}\cdot\frac{1}{4}+\&c.\ \ldots\ (56a),$$

the terms of which, beginning at the head of each column, are as

$$\frac{3}{3},\ \frac{3}{6},\ \frac{3}{10},\ \frac{3}{15},\ \&c.,$$

and may be reduced to one line as follows:—

$$m_3 = \frac{1}{3\,.\,6}\cdot\frac{2}{4} + \frac{3}{6\,.\,10}\cdot\frac{2}{5} + \frac{6}{10\,.\,15}\cdot\frac{2}{6} + \frac{10}{15\,.\,21}\cdot\frac{2}{7} + \&c.\ \ldots\ (57)$$

$$= \frac{1}{6^2} + \frac{2}{10^2} + \frac{3}{15^2} + \frac{4}{21^2} + \frac{5}{28^2} + \frac{6}{36^2} + \&c.\ \ldots\ldots\ldots\ldots\ (58).$$

Now, by taking (40) from (39) in such a way that the first term of the latter is untouched, but its second term is reduced by the first term of (40), and so on, we find that

$$\frac{1}{10^2} + \frac{4}{15^2} + \frac{8}{21^2} + \frac{13}{28^2} + \frac{19}{36^2} + \&c. = \frac{1}{4}\ \ldots\ldots\ldots\ (59).$$

Moreover, by reference to (9), remembering that $r_\infty = 1 + m_2$, we have

$$\frac{1}{3\,.\,6} + \frac{2}{4\,.\,10} + \frac{3}{5\,.\,15} + \&c. = m_2 = \frac{2}{6^2} + \frac{5}{10^2} + \frac{9}{15^2} + \frac{14}{21^2} + \&c.\ldots\ (60).$$

And, by taking (58) from (60), we have

$$m_3 = m_2 - \left(\frac{1}{6^2} + \frac{3}{10^2} + \frac{6}{15^2} + \frac{10}{21^2} + \frac{15}{28^2} + \&c.\right),$$

which, by treatment with (59), becomes

$$m_3 = m_2 - 2\left(\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \frac{1}{15^2} + \frac{1}{21^2} + \&c.\right)\ldots\ldots\ (61).$$

And since, by (45),

$$m_2 = \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \&c.\right),$$

therefore $m_3 = 5m_2 - 2 = \frac{1}{2} - \frac{5}{2}\left(\frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \&c.\right)$ ...... (62)

$= \cdot 101319$ approximately.

On testing our progress by (2), we have given

$$\left.\begin{aligned} a &= \cdot 09375 \\ a^2 &= \cdot 0087890625 \\ a^3 &= \cdot 000823974609375 \\ a^4 &= \cdot 0000772476196289 0625 \\ a^5 &= \cdot 0000072418 \\ a^6 &= \cdot 000000679,\ \&c. \end{aligned}\right\}\ \ldots\ldots\ldots\ldots\ (63)$$

We then find that, using A (= 1) and B (= 2) only, we have

$$\left.\begin{aligned} 4\pi &= \frac{1}{a}\left\{\frac{1 + 2\,.\,2a + \ldots}{1 + \quad 2a + \ldots}\right\} = \frac{1}{a}\cdot\left\{\frac{1\cdot 3750 + \ldots}{1\cdot 1875 + \ldots}\right\} = 12\cdot 350877 + \ldots \\ &= 4\,(3\cdot 087719 + \ldots) \end{aligned}\right\}\ (64);$$

also that, adding C $(= 1 + m_2 = 1{\cdot}4202638)$, we have

$$4\pi = \frac{1}{a}\left\{\frac{1{\cdot}3750 + 3\,(1{\cdot}4202638)\,a^2 + \ldots}{1{\cdot}1875 + \quad (1{\cdot}4202638)\,a^2 + \ldots}\right\} = \frac{1}{a}\left\{\frac{1{\cdot}412448362 + \ldots}{1{\cdot}199827873 + \ldots}\right\}$$

$$= 12{\cdot}555269 + \ldots = 4\,(3\;1388 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (65),$$

and that, further, adding D $(= m_2 + m_3 = {\cdot}5215828)$, we have

$$4\pi = \frac{1}{a}\left\{\frac{1{\cdot}412448362 + 4\,({\cdot}5215828)\,a^3 + \ldots}{1{\cdot}1999827873 + ({\cdot}5215828)\,a^3 + \ldots}\right\}$$

$$= \frac{1}{a}\left\{\frac{1{\cdot}414167446 + \ldots}{1{\cdot}200412558 + \ldots}\right\} = 12{\cdot}56605707 + \ldots$$

$$= 4\,(3{\cdot}14151427 + \ldots) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (66),$$

which is right to four places of decimals

In considering the values of $m_4 \ldots m_n$, we need not write each compound series out in detail, as in (52)—(56), nor even as in (52a)—(56a). There is a law for the formation of the single line exemplified in (57). Using the letters described in (1) and (4), the several terms of this line, in the case of m_n, are as follows:—

$$\left.\begin{array}{ll}
\text{I.} & \dfrac{1}{p_n}\cdot\dfrac{2}{n+1} \\[2ex]
\text{II.} & \dfrac{k_2 + k_3 + \ldots k_n}{p_{n+1}}\cdot\dfrac{2}{n+2} \\[2ex]
\text{III.} & \dfrac{k_2\,(k_3 + \ldots k_{n+1}) + \ldots k_{n-1}\,(k_n + k_{n+1}) + k_n\,.\,k_{n+1}}{p_{n+2}}\cdot\dfrac{2}{n+3} \\[2ex]
\text{IV.} & \dfrac{k_2\,.\,k_3\,(k_4 + \ldots k_{n+2}) + \ldots k_2\,.\,k_{n+1}\,.\,k_{n+2} + \ldots k_n\,.\,k_{n+1}\,.\,k_{n+2}}{p_{n+3}}\cdot\dfrac{2}{n+4} \\[2ex]
\text{V.} & \dfrac{\begin{array}{r} k_2k_3k_4\,(k_5 + \ldots k_{n+3}) + \ldots k_2\,.\,k_{n+1}k_{n+2}k_{n+3} \\ + \ldots k_n\,.\,k_{n+1}k_{n+2}k_{n+3}\end{array}}{p_{n+4}}\cdot\dfrac{2}{n+5}
\end{array}\right\} (67),$$

and so on. It will be seen that all the factors of the denominator (p_{n+x}), except the highest (k_{n+x}), figure in the numerator,—separately in II., in couples in III., in triple's in IV., in quadruplets in V., &c. &c., and in all possible combinations in each several case, but without any repetition.

We can, however, arrange the terms of the several grades of m in columns, according to common denominators (p_{n+x}) and common factors $[2/(n+x+1)]$, the numerators being summed and given in figures.

Common Factors	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{5}$	$\frac{2}{6}$	$\frac{2}{7}$	
Common Denominators	3	3.6	3.6.10	3.6.10.15	3.6.10.15.21	
$m_2 =$	1	3	18	180	2700	+
$m_3 =$		1	9	108	1800	+
$m_4 =$			1	19	393	+
$m_5 =$				1	34	+
$m_6 =$					1	+

(68).

Now, by referring to (29), and bracing the several numerators together, as there shown, we obtain the following:—

$$\left.\begin{array}{l}
C = 1 + m_2 = 1 + \frac{1}{3}\cdot\frac{2}{3} + \frac{3}{3.6}\cdot\frac{2}{4} + \frac{18}{3.6.10}\cdot\frac{2}{5} + \frac{180}{3.6.10.15}\cdot\frac{2}{6} + \frac{2700}{3.6.10.15.21}\cdot\frac{2}{7} + \\
D = m_2 + m_3 = \ldots\ \frac{1}{3}\cdot\frac{2}{3} + \frac{4}{3.6}\cdot\frac{2}{4} + \frac{27}{3.6.10}\cdot\frac{2}{5} + \frac{288}{3.6.10.15}\cdot\frac{2}{6} + \frac{4500}{3.6.10.15.21}\cdot\frac{2}{7} + \\
E = m_3 + m_4 = \ldots\ \ldots\ \frac{1}{3.6}\cdot\frac{2}{4} + \frac{10}{3.6.10}\cdot\frac{2}{5} + \frac{127}{3.6.10.15}\cdot\frac{2}{6} + \frac{2193}{3.6.10.15.21}\cdot\frac{2}{7} + \\
F = m_4 + m_5 = \ldots\ \ldots\ \frac{1}{3.6.10}\cdot\frac{2}{5} + \frac{20}{3.6.10.15}\cdot\frac{2}{6} + \frac{427}{3.6.10.15.21}\cdot\frac{2}{7} + \\
G = m_5 + m_6 = \ldots\ \ldots\ \ldots\ \ldots\ \frac{1}{3.6.10.15}\cdot\frac{2}{6} + \frac{35}{3.6.10.15.21}\cdot\frac{2}{7} +
\end{array}\right\} (69),$$

and so on; from which it will be seen that the numerators, excluding the common factors, follow the order given in (3), from A for C, from B for D, from C for E, &c.; and can therefore be written down with ease. Moreover, when this is done to any required extent, but little further trouble is needed to split up the numerators to suit the respective grades of m; it is a mere matter of subtraction, as can easily be seen. But, if thought desirable, they can be found by Algebra from the formulæ $x - 3y + 3^2z - \&c.$, $x - 6y + 6^2z - \&c.$, $x - 10y + 10^2z - \&c.$—&c., as found in (30)—(32), taking care to use one less equation (beginning from the first) than the numerators in a column drawn up according to (68), the lowest numerator being known to be 1. Thus, from the four equations,

$$x - 3y + 9z - 27u + 81 = 0, \quad x - 6y + 36z - 216u + 1296 = 0,$$
$$x - 10y + 100z - 1000u + 10000 = 0, \quad x - 15y + 225z - 3375u + 50625 = 0,$$

we obtain $u = 34$, $z = 393$, $y = 1800$, $x = 2700$, the numerators in the last column given of (68).

The first twelve terms of E amount to ·1051126, and there are indications that it would reach ·116 if carried on indefinitely, since it converges slowly. The first eleven terms of F amount to ·0139405, and it would probably reach ·017, for it converges still more slowly than E. But the two series do not yield readily to summation by methods not presupposing π as a known quantity, nor even with it. It is highly probable, however, that the above values are nearly correct. Such being the case,

$$4\pi = \frac{1}{a}\left\{\frac{1{\cdot}414167446 + 5\,({\cdot}116)\,a^4 + 6\,({\cdot}017)\,a^5 + \ldots}{1{\cdot}200412558 + ({\cdot}116)\,a^4 + ({\cdot}017)\,a^5 + \ldots}\right\}$$

$$= \frac{1}{a}\left\{\frac{1{\cdot}414212988 + \ldots}{1{\cdot}200421642 + \ldots}\right\} = 12{\cdot}5663666 + \ldots = 4\,(3{\cdot}14159 + \ldots).$$

The veteran and world-famed mathematician, Mr. W. S. B. Woolhouse, F.R.A.S., who refers to an article of his in the *Ladies' Diary* of 1836, has supplied the following valuable information as to the summation of intricate series of the kind treated of in the foregoing paper.

"Respecting series whose terms are the product of the reciprocals of sequent polygonal numbers, the following method may generally be applied:—Let f characterise a function, and let

$$\Sigma f(x) = f(1) + f(2) \ldots + f(n) + f(n+1) + \&c.,$$
$$\Sigma f(x+n) = \qquad\qquad f(n+1) + \&c.$$

Then $$\Sigma\left\{f(x)-f(x+n)\right\} = \Sigma_1^c f(c),$$
so that all functions of the form
$$F(x) = f(x)-f(x+n) \quad\text{.........................}\quad (A)$$
can thus be summed to infinity. It is therefore only necessary to decompose a function into terms of the form $F(x)$ with or without other terms the sums of which are previously known.

Referring to the particular series [given above as that of m_2, *vide* (42)],
$$\frac{1}{1.3}+\frac{1}{3.6}+\frac{1}{6.10}+\frac{1}{10.15}+\&c.$$
$$=\Sigma\frac{1}{\frac{1}{2}x(x+1)\,.\,\frac{1}{2}(x+1)(x+2)}=\Sigma\frac{4}{x(x+1)^2(x+2)}$$
$$=4\Sigma\left(\frac{1}{x}-\frac{1}{x+1}\right)\left(\frac{1}{x+1}-\frac{1}{x+2}\right)$$
$$=4\Sigma\left\{\frac{1}{x(x+1)}+\frac{1}{(x+1)(x+2)}-\frac{1}{x(x+2)}-\frac{1}{(x+1)^2}\right\}$$
$$=4\Sigma\left\{\left(\frac{1}{x}-\frac{1}{x+1}\right)+\left(\frac{1}{x+1}-\frac{1}{x+2}\right)-\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)-\frac{1}{(x+1)^2}\right\}$$
$$=4\Sigma\left\{\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)-\frac{1}{(x+1)^2}\right\}.$$
Here $\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)$ is of the form $F(x)$, and, by (A), the sum is
$$\Sigma\left\{\frac{1}{2}\left(\frac{1}{x}-\frac{1}{x+2}\right)\right\}=\frac{1}{2}\left(1+\frac{1}{2}\right)=\frac{3}{4}.$$
The last term, $-\Sigma[1/(x+1)^2]$ is a known form, and the sum is $(\pi^2/6)-1$. The sum of the series in question is therefore
$$4\left\{\frac{3}{4}-\left(\frac{\pi^2}{6}-1\right)\right\}=7-\frac{2}{3}\pi^2.$$
The same method applies easily to the other series (m_3), namely,
$$\frac{1}{3.6}\cdot\frac{2}{4}+\frac{3}{6.10}\cdot\frac{2}{5}+\frac{6}{10.15}\cdot\frac{2}{6}+\frac{10}{15.21}\cdot\frac{2}{7}+\&c.$$
$$=\frac{1}{6^2}+\frac{2}{10^2}+\frac{3}{15^2}+\frac{4}{21^2}+\&c.$$
Here the x^{th} term is $4x/\left\{(x+2)(x+3)\right\}^2$, and this expression, when decomposed, becomes
$$20\left(\frac{1}{x+2}-\frac{1}{x+3}\right)-\frac{8}{(x+2)^2}-\frac{12}{(x+3)^2}.$$
The required sum is therefore
$$20\Sigma\left(\frac{1}{x+2}-\frac{1}{x+3}\right)-8\Sigma\frac{1}{(x+2)^2}-\Sigma\frac{12}{(x+3)^2}$$
$$=\frac{20}{3}-8\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)-12\left(\frac{\pi^2}{6}-1-\frac{1}{4}-\frac{1}{9}\right)=33-\frac{10}{3}\pi^2,$$

The process may evidently be extended to any number of factors, and the result in every case must be of the form $A - B\pi^2$. Thus

$$\frac{1}{1.3.6}+\frac{1}{3.6.10}+\frac{1}{6.10.15}+\&c. = \Sigma\frac{8}{x(x+1)^2(x+2)^2(x+3)},$$

$$\frac{1}{1.3.6.10}+\frac{1}{3.6.10.15}+\frac{1}{6.10.15.21}+\&c.$$

$$=\Sigma\frac{16}{x(x+1)^2(x+2)^2(x+3)^2(x+4)}.$$

The x^{th} terms all admit of being algebraically decomposed into separate fractions. Thus

$$\Sigma\frac{8}{x(x+1)^2(x+2)^2(x+3)}=\Sigma\left\{\frac{2}{3}\left(\frac{1}{x}-\frac{1}{x+3}\right)+6\left(\frac{1}{x+1}-\frac{1}{x+2}\right)\right.$$
$$\left.-4\left(\frac{1}{(x+1)^2}+\frac{1}{(x+2)^2}\right)\right\}$$

$$=\frac{2}{3}\left(1+\frac{1}{2}+\frac{1}{3}\right)+6\left(\frac{1}{2}\right)-4\left(\frac{\pi^2}{6}-1+\frac{\pi^2}{6}-1-\frac{1}{4}\right)=\frac{119}{9}-\frac{4}{3}\pi^2;$$

$$\Sigma\frac{16}{x(x+1)^2(x+2)^2(x+3)^2(x+4)}=\Sigma\left\{\frac{1}{9}\left(\frac{1}{x}-\frac{1}{x+4}\right)+\frac{28}{9}\left(\frac{1}{x+1}-\frac{1}{x+3}\right)\right.$$
$$\left.-\frac{4}{3}\left(\frac{1}{(x+1)^2}+\frac{1}{(x+3)^2}\right)-4\left(\frac{1}{(x+2)^2}\right)\right\}$$

$$=\frac{1}{9}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{28}{9}\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{4}{3}\left(\frac{\pi^2}{6}-1+\frac{\pi^2}{6}-1-\frac{1}{4}-\frac{1}{9}\right)$$
$$-4\left(\frac{\pi^2}{6}-1-\frac{1}{4}\right)=\frac{395}{36}-\frac{10}{9}\pi^2."$$

I have not been able to find anything like this in any of the ordinary text-books, though *Gross's Algebra*, Part II., comes nearest to it, so far as I can judge. It has been of essential service in verifying my results, which, however, were bound to be obtained independently, as π could not be presupposed in an investigation regarding its own value.

Mr. Woolhouse also supplies the following as to the summation of two sets of series:—

"The expression $u_n = 1/\{x^2 \dots (x+n)^2\}$, may be partially decomposed in the following manner.

$$u_n = A\left\{\frac{1}{x^2\dots(x+n-1)^2}+\frac{1}{(x+1)^2\dots(x+n)^2}\right\}$$
$$+B\left\{\frac{2}{(x+1)^2\dots(x+n-1)^2}-\frac{1}{x^2\dots(x+n-2)^2}-\frac{1}{(x+2)^2\dots(x+n)^2}\right\}$$
$$=\frac{2n-1}{n^3}\left\{\frac{1}{x^2\dots(x+n-1)^2}+\frac{1}{(x+1)^2\dots(x+n)^2}\right\}$$
$$+\frac{1}{n^3(n-1)}\left\{\frac{2}{(x+1)^2\dots(x+n-1)^2}-\frac{1}{x^2\dots(x+n-2)^2}-\frac{1}{(x+2)^2\dots(x+n)^2}\right\}.$$

Let $U_n = \Sigma u_n$ when x takes the values 1, 2, ... ∞ ; then, from what precedes,

$$U_n = \frac{2n-1}{n^3}\left\{U_{n-1} + \left(U_{n-1} - \frac{1}{(n\,!)^2}\right)\right\}$$

$$+\frac{1}{n^3(n-1)}\left\{\left(2U_{n-2} - \frac{2}{(n-1\,!)^2}\right) - U_{n-2} - \left(U_{n-2} - \frac{1}{(n-1\,!)^2} - \frac{1}{(n\,!)^2}\right)\right\}$$

$$= \frac{4n-2}{n^3} \cdot U_{n-1} - \frac{2n-1}{n^3(n\,!)^2} - \frac{1}{n^3(n-1)}\left\{\frac{1}{(n-1\,!)^2} - \frac{1}{(n\,!)^2}\right\}$$

$$= \frac{4n-2}{n^3} \cdot U_{n-1} - \frac{2n-1}{n^3(n\,!)^2} - \frac{1}{n^3(n-1)} \cdot \frac{1}{(n!)^2}(n^2-1)$$

$$= \frac{4n-2}{n^3} \cdot U_{n-1} - \frac{1}{n^3(n\,!)^2}\left\{(2n-1)+(n+1)\right\}$$

$$= \frac{4n-2}{n^3} \cdot U_{n-1} - \frac{3}{n^2(n\,!)^2}$$

$$= \frac{1}{n^2}\left\{\frac{4n-2}{n} \cdot U_{n-1} - \frac{3}{(n\,!)^2}\right\} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (a).$$

From this relation, beginning with $U_0 = \pi^2/6$, we obtain, successively,

$$U_0 = \frac{\pi^2}{6}, \qquad U_1 = \frac{\pi^2}{3} - 3,$$

$$U_2 = \frac{\pi^2}{4} - \frac{39}{16}, \qquad U_3 = \frac{5\pi^2}{54} - \frac{197}{216},$$

$$U_4 = \frac{35\pi^2}{1728} - \frac{5525}{27648}, \qquad U_5 = \frac{7\pi^2}{2400} - \frac{9211}{320000},$$

&c. &c.

$$\frac{1}{1\,.\,3\,.\,6 \text{ to } n \text{ factors}} = \frac{1}{\frac{1}{2}x(x+1)\,.\,\frac{1}{2}(x+1)(x+2)\ldots\frac{1}{2}(x+n-1)(x+n)}$$

$$= \frac{2^n}{x\left\{(x+1)^2\ldots(x+n-1)^2\right\}(x+n)}.$$

Let
$$v_n = \frac{1}{x\left\{(x+1)^2\ldots(x+n-1)^2\right\}(x+n)}$$

$$= \frac{x(x+n)}{x^2\ldots(x+n)^2} = \frac{\frac{1}{2}\left\{(x+n)^2+x^2-n^2\right\}}{x^2\ldots(x+n)^2}$$

$$= \frac{1}{2}\left\{\frac{1}{x^2\ldots(x+n-1)^2} + \frac{1}{(x+1)^2\ldots(x+n)^2} - \frac{n^2}{x^2\ldots(x+n)^2}\right\}.$$

Hence, by summation,

$$V_n = \frac{1}{2}\left\{U_{n-1} + \left(U_{n-1} - \frac{1}{(n\,!)^2}\right) - n^2 U_n\right\};$$

that is, substituting for U_n from (a),

$$V_n = \frac{1}{(n\,!)^2} - \frac{n-1}{n} \cdot U_{n-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (\beta),$$

which is a convenient relation.

By using the preceding values of U, we find, from (β),

$$V_1 = 1, \qquad V_2 = \frac{7}{4} - \frac{\pi^2}{6},$$

$$V_3 = \frac{119}{72} - \frac{\pi^2}{6}, \qquad V_4 = \frac{395}{576} - \frac{5\pi^2}{72},$$

$$V_5 = \frac{27637}{172800} - \frac{7\pi^2}{432}, \qquad V_6 = \frac{248717}{10368000} - \frac{7\pi^2}{2880}.$$

&c. &c.

By multiplying these respectively by 2, 2^2, 2^3, &c., we find

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6}, \text{ \&c.} = 2,$$

$$\frac{1}{1.3} + \frac{1}{3.6} + \frac{1}{6.10}, \text{ \&c.} = 7 - \frac{2\pi^2}{3},$$

$$\frac{1}{1.3.6} + \frac{1}{3.6.10} + \frac{1}{6.10.15}, \text{ \&c.} = \frac{119}{9} - \frac{4\pi^2}{3},$$

$$\frac{1}{1.3.6.10} + \frac{1}{3.6.10.15}, \text{ \&c.} = \frac{395}{36} - \frac{10\pi^2}{9},$$

$$\frac{1}{1.3.6.10.15} + \frac{1}{3.6.10.15.21}, \text{ \&c.} = \frac{27637}{5400} - \frac{14\pi^2}{27},$$

$$\frac{1}{1.3.6.10.15.21}, \text{ \&c.} = \frac{248717}{162000} - \frac{7\pi^2}{45}.$$

&c. &c.

These results may all be readily extended by means of (α) and (β).

It may also be noted that the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \text{\&c.} = \Sigma \frac{1}{\left(x\dfrac{x+1}{2}\right)^2} = \Sigma \frac{4}{x^2(x+1)^2} = 4U_1 = \tfrac{4}{3}\pi^2 - 12.$$

$$\therefore \quad \frac{1}{3^2} + \frac{1}{6^2} + \frac{1}{10^2} + \text{\&c.} = \tfrac{4}{3}\pi^2 - 13;$$

and, by (62), $\qquad m_2 = \frac{1}{2} - \frac{1}{2}(\frac{4}{3}\pi^2 - 13) = 7 - \frac{2}{3}\pi^2,$

$$m_3 = 5m_2 - 2 = 33 - \tfrac{10}{3}\pi^2.\text{''}$$

APPENDIX II.

SOLUTIONS TO UNSOLVED QUESTIONS.

BY JAMES McMAHON, B.A.

7394. (W. J. C. SHARP, M.A.)—If a quartic equation represent four straight lines, its Hessian represents the same four lines and an imaginary conic. Explain the geometrical signification of this.

Solution.

This quartic equation can be put in the form

$$xyz\,(x+y+z) = 0.$$

Its Hessian will then take the form

$$\begin{vmatrix} 2yz, & 2xz+2yz+z^2, & 2xy+y^2+2yz \\ 2xz+2yz+z^2, & 2xz, & x^2+2xy+2xz \\ 2xy+y^2+2yz, & x^2+2xy+2xz, & 2xy \end{vmatrix} = 0.$$

This determinant vanishes with x; therefore x is a factor. Similarly, y and z are factors, and (since the Hessian bears a symmetrical relation to the four lines) $x+y+z$ must also be a factor; the fourth factor is (by symmetry) of the form

$$a\,(x^2+y^2+z^2)+b\,(xy+xz+yz),$$

and, by giving special values to x, y, z, we find $a = b = 6$. Hence the Hessian consists of the four lines and the imaginary conic

$$(y+z)^2+(z+x)^2+(x+y)^2 = 0.$$

To examine some of the geometrical properties of this system, it will be remembered that, if the first polar of any point P with regard to an algebraic curve have a double point Q, then the polar conic of Q will have a double point P; that the locus of Q is the Hessian, and of P the Steinerian.

Now, if the coordinates of P be x', y', z', its first polar as to the given quartic will be

$$xyz\,(x'+y'+z')+(x+y+z)\,(x'yz+y'xz+z'xy) = 0,$$

which may be written in the form

$$xyzu'+u\,(x'yz+y'xz+z'xy) = 0,$$

and is a cubic passing through the six intersections of the four lines. If P lie on one of these lines (x say), then $x' = 0$, and the cubic breaks up into x and a conic; it has therefore double points lying on x. Hence this line must appear as a factor in the equations of both Steinerian and Hessian; similarly for each of the other lines. It may be verified,

conversely, that the polar conic of any point on x has a double point lying on x. Again, if P be at the intersection of two of the lines (xy say), its first polar will break up into x, y, and a line through (zu) harmonic conjugate to the line joining (xy) to (zu). It appears from the equation of the first polar that it does not break up unless P is on one of the four lines. But, when P is not on one of these lines, its first polar cannot have a double point on these lines, and hence can only have a double point Q on the imaginary conic; in this case the polar conic of Q must be imaginary, and break up into a pair of lines intersecting in an imaginary point P on the Steinerian. Hence the only real points of the Steinerian are on the four given lines. And we have seen that the corresponding double points on first polars are also on these lines; thus the imaginary conic of the Hessian corresponds to the imaginary (eighth degree) part of the Steinerian.

It may be added that the Cayleyan, which is the envelope of the line PQ, must contain the six intersections of the four given lines as factors of its tangential equation.

7847. (W. J. C. Sharp, M.A.)—If we denote the operation

$$l\frac{d}{dx}+m\frac{d}{dy}+n\frac{d}{dz},$$

where l, m, n are direction cosines, by $\frac{d}{dh}$, so that $\frac{d}{dh}$ is the differential coefficient with respect to an axis (Clerk Maxwell's *Electricity*, 2nd Ed., p. 180); show that it is an invariant symbol of operation.

Solution.

Let $V = C$ be the equation of any equipotential surface; then

$$\frac{dV}{dx},\ \frac{dV}{dy},\ \frac{dV}{dz}$$

are the components, parallel to the coordinate axes, of the forces acting at x, y, z, and

$$l\frac{dV}{dx}+m\frac{dV}{dy}+n\frac{dV}{dz}$$

is the component, parallel to the fixed line, whose direction cosines are l, m, n; it must therefore be independent of the position of the coordinate axes. This can be easily verified by transformation of coordinates, for

when $$x = a+l_1X+l_2Y+l_3Z, \text{ \&c.},$$

then $$l = l_1L+l_2M+l_3N, \text{ \&c.}$$

and $$\frac{dV}{dx}=l_1\frac{dV}{dX}+l_2\frac{dV}{dY}+l_3\frac{dV}{dZ}, \text{ \&c.};$$

therefore $$l\frac{dV}{dx}+m\frac{dV}{dy}+n\frac{dV}{dz}=L\frac{dV}{dX}+M\frac{dV}{dY}+N\frac{dV}{dZ}.$$

[The Proposer's solution is given in a paper "On the Invariants of a certain Orthogonal Transformation" in the *Proceedings of the London Mathematical Society*, Vol. xiii., Art. ii., p. 222.]

8832. (Professor MAHENDRA NATH RAY, M.A., LL.B.)—A particle P moves in a conic section under the action of a force tending to one focus S. If the tangent to the conic section at P intersect the directrix corresponding to S in the point D, show that the angular velocities of P and D about S are equal, and that the velocity of D varies inversely as the square of the ordinate of P.

Solution.

Since PSD is constant (a right angle), the angular velocities of P and D about S are equal.

Again, let SX, ($=q$), be the perpendicular from S to the directrix, and p the perpendicular from S to PD; let SP $= r$, XSP $= \theta$, SPD $= \phi$, SD $= \rho$; let δs and $\delta\sigma$ be the respective infinitesimal distances passed over by P and D, in the time δt. Then

$$p\,\delta s \doteq r^2\,\delta\theta, \quad q\,\delta s \doteq \rho^2\,\delta\theta;$$

therefore $$q\,\frac{d\sigma}{dt}\Big/p\,\frac{ds}{dt} = \rho^2/r^2 = \tan^2\phi = \left(r\,\frac{d\theta}{dr}\right)^2.$$

But since $r \propto 1/1+e\cos\theta$, therefore $r\,(d\theta/dr) \propto 1/r\sin\theta$;

and since $p\,(ds/dt)$ is constant, being twice the rate at which SP describes areas around the centre of force,

therefore $$d\sigma/dt \propto 1/r^2\sin^2\theta.$$

Therefore, &c.

8939. (W. J. C. SHARP, M.A.)—If

$$S \equiv ax^2+by^2+cz^2+dw^2+2lyz+2mzx+2nxy+2pxw+2qyw+2rzw = 0$$

be the equation to a quadric, show that (1) the line joining the points (x_1, y_1, z_1, w_1), (x_2, y_2, z_2, w_2) will meet the surface in two real, two coincident, or two imaginary points, according as $S_1S_2-P_{12}^2$ is positive, zero or negative, where $P_{12}=0$ is the condition that one of the points should, lie on the polar plane of the other; and (2) express the same condition in terms of the tangential equation to the quadric.

Solution.

(1) The coordinates of any point on the line will be of the form $\lambda x_1+\mu x_2$, &c. Substituting in the equation of the surface, we get $\lambda^2 S_1+2\lambda\mu P_{12}+\mu^2 S_2 = 0$, which gives the two values of λ/μ corresponding to the two intersections. The roots are real, coincident, or imaginary, according as $S_1S_2-P_{12}^2$ is positive, zero, or negative; therefore, &c.

(2) To express $S_1S_2-P_{12}^2$ in terms of the coefficients of the tangential equation $\Sigma = 0$, let us first find, in terms of these coefficients, the condition that the line joining the given points may touch the surface.

The tangent plane $\alpha x+\beta y+\gamma z+\delta w = 0$ touches the surface in the point whose coordinates are $d\Sigma/d\alpha$, $d\Sigma/d\beta$, &c.;

therefore we have the equations

$$A\alpha+H\beta+G\gamma+L\delta+\lambda x_1+\mu x_2 = 0,$$
$$H\alpha+B\beta+F\gamma+M\delta+\lambda y_1+\mu y_2 = 0,$$

$$\begin{aligned}
&G\alpha + F\beta + C\gamma + N\delta + \lambda z_1 + \mu z_2 = 0,\\
&L\alpha + M\beta + N\gamma + D\delta + \lambda w_1 + \mu w_2 = 0,\\
&x_1\alpha + y_1\beta + z_1\gamma + w_1\delta = 0,\\
&x_2\alpha + y_2\beta + z_2\gamma + w_2\delta = 0.
\end{aligned}$$

And, eliminating α, β, γ, δ, λ, μ, we have the condition

$$\begin{vmatrix} A & H & G & L & x_1 & x_2 \\ H & B & F & M & y_1 & y_2 \\ G & F & C & N & z_1 & z_2 \\ L & M & N & D & w_1 & w_2 \\ x_1 & y_1 & z_1 & w_1 & 0 & 0 \\ x_2 & y_2 & z_2 & w_2 & 0 & 0 \end{vmatrix} = 0.$$

Now this determinant can differ only by a factor from $S_1S_2 - P_{12}^2$, and it will be seen on comparing coefficients that this factor is the discriminant, which is the cube root of the determinant

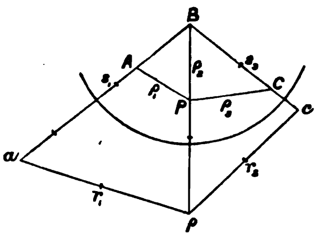
$$\begin{vmatrix} A & H & G & L \\ H & B & F & M \\ G & F & C & N \\ L & M & N & D \end{vmatrix}.$$

Hence $S_1S_2 - P_{12}^2$ is expressed in terms of the tangential equation to the quadric.

It may be added that the determinant equation above is the equation of the tangent cone whose vertex is at (x_1, y_1, z_1, w_1), if x_2, y_2, z_2, w_2 be regarded as current coordinates.

9432. (The EDITOR.)—Find the locus of a point whose distance from one of three given points is (1) an arithmetic, (2) a geometric, (3) an harmonic mean, between its distances from the other two.

Solution.



1. Let the distances of P from A, B, C be such that $\rho_1 + \rho_3 = 2\rho_2$. Let p, a, c be inverse to P, A, C with respect to B as centre, and let r_1, r_3, s_1, s_3 stand for pa, pc, Ba, Bc. Then we have, by similar triangles,

$$r_1/s_1 = \rho_1/\rho_2, \quad r_3/s_3 = \rho_3/\rho_2;$$

therefore, $r_1/s_1 + r_3/s_3 = 2$,

therefore p describes a Cartesian with a, c as foci, and P describes the inverse of this oval with respect to B.

2. We have now $\rho_1\rho_3 = \rho_2^2$, therefore, $r_1 r_3 = s_1 s_3$;
hence p describes a Cassinian oval, and P describes its inverse.

3. In this case, $1/\rho_1 + 1/\rho_3 = 2/\rho_2$,
hence p describes the oval $s_1/r_1 + s_3/r_3 = 2$, and P describes its inverse with respect to B.

9473. (J. BRILL, M.A.)—Two families of equipotential curves are traced on a plane. Another family of curves is drawn, each curve of which possesses the property that the tangent at any point of it divides in a constant ratio the angle between the tangents at that point of the particular curves of the other two families that intersect in the said point; prove that this last family is also an equipotential system. Extend the theorem to suit the case in which we have three or more families of equipotential curves traced on the plane.

Solution.

Let the equation of the first family of equipotential curves be $f(x, y) = c$, a variable parameter; then the particular curve of this family traced through (x_1, y_1) is $f(x, y) = f(x_1, y_1)$, and its tangent at (x_1, y_1) is

$$L \equiv (x-x_1)\left(\frac{df}{dx}\right)_1 + (y-y_1)\left(\frac{df}{dy}\right)_1 = 0;$$

similarly the tangent at (x_1, y_1) to the curve of the second family is

$$M \equiv (x-x_1)\left(\frac{d\phi}{dx}\right)_1 + (y-y_1)\left(\frac{d\phi}{dy}\right)_1 = 0;$$

and it is evident that the line $L + \kappa M = 0$ touches at (x_1, y_1) the particular curve of the family $f + \kappa\phi = e$ that passes through (x_1, y_1).

In the same way a family $f + \kappa\phi + \lambda\psi = h$ may be obtained.

7899. (W. J. C. SHARP, M.A.)—If lines be drawn from the vertices of a triangle through any point on a circumscribed conic to meet the opposite sides; prove that the axis of homology of the triangle formed by the connectors of the intersections and the original triangle will pass through a fixed point, which is on each of the fourth harmonics to the tangents at the vertices.

9130. (W. J. C. SHARP, M.A.)—If lines be drawn through any point on the circumcircle of a triangle to meet the opposite sides, the axis of homology of the triangle formed by joining these points, and of the original triangle, will pass through a fixed point; define this point. Also show that the proposition is true for other circumscribed conics, and state the analogous proposition in the geometry of space of n dimensions.

Solution.

Let the given triangle be the triangle of reference; then the equation of any circumscribed conic can be put in the form

$$fyz + gzx + hxy = 0, \text{ or } f/x + g/y + h/z = 0.$$

It is well known that the axis of homology corresponding to any point $(x'y'z')$ is $\quad x/x' + y/y' + z/z' = 0;$

but, when $(x'y'z')$ lies on the given circumscribed conic, we have

$$f/x' + g/y' + h/z' = 0,$$

therefore the axis of homology passes through the fixed point (fgh).

Again, the tangent at $(1, 0, 0)$ is $y/g + z/h = 0$, and its fourth harmonic is $y/g - z/h = 0$, which passes through the point (fgh); therefore this point lies on each of the fourth harmonics to the tangents at the vertices.

In three dimensions, any cubic surface through the six edges of the tetrahedron of reference can be represented by

$$a/x + b/y + c/z + d/u = 0.$$

Any point $(x'y'z'u')$ has a plane of homology whose equation is

$$x/x' + y/y' + z/z' + u/u' = 0;$$

for, if we draw lines from the vertices through $(x'y'z'u')$, they will meet the opposite faces in the points $(0\,y'z'u')$, $(x'\,0\,z'u')$, &c., which determine four planes whose equations are

$$\begin{vmatrix} x & y & z & u \\ 0 & y' & z' & u' \\ x' & 0 & z' & u' \\ x' & y' & 0 & u' \end{vmatrix} = 0, \text{ \&c.,}$$

or $$x/x' + y/y' + z/z' - 2u/u' = 0, \text{ \&c.,}$$

and these meet the opposite planes $u = 0$, &c. in four lines, all lying in the plane $$x/x' + y/y' + z/z' + u/u' = 0.$$
Now, if the point $(x'y'z'u')$ lie on the given cubic surface, we have

$$a/x' + b/y' + c/z' + d/u' = 0;$$

therefore the plane of homology must pass through the fixed point $(abcd)$.

In four dimensions, any fourth degree surface through the edges of the "simplicissimum" of reference can be represented by

$$a/x + b/y + c/z + d/u + e/v = 0.$$

Lines through the vertices and the point $(x'y'z'u'v')$ will meet the opposite faces in five points, four of which will lie in the hyper-plane

$$\begin{vmatrix} x & y & z & u & v \\ 0 & y' & z' & u' & v' \\ x' & 0 & z' & u' & v' \\ x' & y' & 0 & u' & v' \\ x' & y' & z' & 0 & v' \end{vmatrix} = 0,$$

or $$x/x' + y/y' + z/z' + u/u' - 3v/v' = 0;$$

we conclude, as before, that the five planes will meet the respective opposite faces of the simplicissimum in five lines that lie in the hyper-plane

$$x/x' + y/y' + z/z' + u/u' + v/v' = 0,$$

and that this plane will pass through the fixed point $(abcde)$ when $(x'y'z'u'v')$ moves on the given fourth-degree surface.

Similarly for any number of dimensions.

8475. (Professor MATHEWS, M.A.) — The biquadratic form $(A_0 A_1 \ldots A_4 \between xy)^4$ may be written symbolically $(a_0 x^2 + 2a_1 xy + a_2 y^2)^2$; viz., in the expansion of the latter, we replace any coefficient $a_i a_k$ by A_{i+k}. If (b_0, b_1, b_2), (c_0, c_1, c_2) ...... are sets of symbols equivalent to (a_0, a_1, a_2), prove that

$$(a_0 b_2 + a_2 b_0 - 2a_1 b_1)^2 = 2\,(A_0 A_4 - 4A_1 A_3 + 3A_2^2) \ldots\ldots\ldots\ldots\ldots (1),$$

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 \end{vmatrix}^2 = 6 \begin{vmatrix} A_0 & A_1 & A_2 \\ A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Solution.

The binary n-ic may be conveniently written in the symbolic form $(a_0x+a_1y)^n$, provided that after expansion we consider a_ia_k equivalent to A_{i+k}, and in consequence $(a_i)^n$ to A_{ni}. It follows that the biquadratic $(a_0x+a_1y)^4$ must be the symbolic square of the quadratic $(a_0x+a_1y)^2$. We can by this method derive certain invariants. Starting with the Jacobian $a_0b_1-a_1b_0$ of two linear functions a_0x+a_1y, b_0x+b_1y, and forming its symbolic square, we obtain $A_0B_2+A_2B_0-2A_1B_1$, an invariant of the system of two quadratics; making the B's equal to the A's, we have $2(A_0A_2-A_1^2)$, an invariant of a single quadratic. Again, we can prove equation (1), and so derive an invariant of the biquadratic, either from $(a_0b_2+a_2b_0-2a_1b_1)^2$ or from $(a_0b_1-a_1b_0)^4$; we get in either case

$$A_0B_4-4A_1B_3+6A_2B_2-4A_3B_1+A_4B_0,$$

which, when we make the B's equal to the A's, becomes

$$2(A_0A_4-4A_1A_3+3A_2^2).$$

Another invariant of the biquadratic is derived in equation (2) by taking the symbolic square of the invariant of the system of three quadratics. The latter invariant may be written

$$a_2(b_0c_1-b_1c_0)+b_2(c_0a_1-c_1a_0)+c_2(a_0b_1-a_1b_0),$$

and the symbolic square of the first term is $A_4(B_0C_2+B_2C_0-2B_1C_1)$, which, when the B's and C's are made equal to A's, becomes $2A_4(A_0A_2-A_1^2)$; the squares of the other two terms reduce to the same expression; again, the double product of the first and second term is

$$2a_2b_2(a_1b_0C_1-a_0b_0C_2-a_1b_1C_0+a_0b_1C_1),$$

or $$2(A_3B_2C_1-A_2B_2C_2-A_3B_3C_0+A_2B_3C_1),$$

which becomes $$2A_3(A_1A_2-A_0A_3)+2A_2(A_1A_3-A_2^2);$$

hence the complete symbolic square of the invariant is equal to

$$6A_4\begin{vmatrix}A_0 & A_1\\ A_1 & A_2\end{vmatrix}+6A_3\begin{vmatrix}A_2 & A_3\\ A_0 & A_1\end{vmatrix}+6A_2\begin{vmatrix}A_1 & A_2\\ A_2 & A_3\end{vmatrix};$$

therefore, &c.

8589. (Professor NEUBERG.)—Trouver la trajectoire orthogonale de toutes les conchoïdes de Nicomède qui ont même pôle et même directrice.

Solution.

The equation of the given system of conchoids is $r=a\sec\theta+b$, in which b is a varying parameter. Let ϕ be the angle between the radius vector and the tangent at any point (r,θ); ϕ' the corresponding angle for the orthogonal curve through the same point. Then

$$\tan\phi'=-\cot\phi=-dr/r\,d\theta=-a\sin\theta/r\cos^2\theta;$$

hence for the orthogonal curve we have

$$r\,d\theta/dr=-a\sin\theta/r\cos^2\theta,$$

therefore $$dr/r^2+\cos^2\theta\,d\theta/a\sin\theta=0,$$

therefore $$a/r-a\cos\theta+\log\tan\tfrac{1}{2}\theta=\kappa.$$

Since this equation does not involve b, it represents a curve orthogonal to every member of the given system; hence, if κ be regarded as a varying parameter, we have a second system of curves each of which is an orthogonal trajectory to the first system.

It may be added that, if the conditions of the question be altered so as to make a the varying parameter, then a (instead of b) must be eliminated from the differential equation, and the new system of orthogonal trajectories would be represented by

$$dr/r^2 + \cos\theta\, d\theta/(r-b)\sin\theta = 0,$$

or $$b/r + \log(r\sin\theta) = \kappa.$$

8959. (W. J. C. Sharp, M.A.)—If any function of the coefficients of the ternary quantic

$$a_0x^n + n(a_1y + b_1z)x^{n-1} + \tfrac{1}{2}n(n-1)(a_2y^2 + 2b_2yz + c_2z^2)x^{n-2} + \&c.$$

vanish when α^n, $\alpha^{n-1}\beta$, $\alpha^{n-1}\gamma$, $\alpha^{n-2}\beta^2$, &c. are substituted for a_0, a_1, b_1, a_2, &c., the evectant of the function will vanish, and conversely.

Solution.

The evectant of a function ϕ of the coefficients of a given ternary quantic U may be briefly defined as follows:—Take the same function ϕ of the coefficients of the quantic $U + k(\alpha x + \beta y + \gamma z)^n$, and arrange the result in ascending powers of k. The coefficients of the successive powers of k are called the successive evectants of the function ϕ: they are contravariant when ϕ is invariant; in other words, they are invariant functions of the coefficients of the system composed of the quantic U and the linear function $\alpha x + \beta y + \gamma z$.

It is easily seen, by Taylor's theorem, that the first evectant is

$$\frac{d\phi}{da_0}\alpha^n + \left(\frac{d\phi}{da_1}\beta + \frac{d\phi}{db_1}\gamma\right)\alpha^{n-1} + \left(\frac{d\phi}{da_2}\beta^2 + \frac{d\phi}{db_2}\beta\gamma + \frac{d\phi}{dc_2}\gamma^2\right)\alpha^{n-2} + \ldots$$

Now, if in ϕ (supposed homogeneous), and in its evectant, we make

$$\alpha^n = a_0, \quad \alpha^{n-1}\beta = a_1, \quad \alpha^{n-1}\gamma = b_1, \ \&c.,$$

it follows, by Euler's theorem, that the evectant will become $p\phi$, where p is the degree of ϕ; hence the two results will vanish together, $\therefore$ &c.

A more direct proof is as follows:—The substitution in question evidently brings U to the form $(\alpha x + \beta y + \gamma z)^n$, and $U + k(\alpha x + \beta y + \gamma z)^n$ to the form $(1+k)(\alpha x + \beta y + \gamma z)^n$; hence any homogeneous function ϕ of the coefficients of the latter quantic can differ from the same function of the coefficients of U only by the multiplier $(1+k)^p$; therefore the successive coefficients of the powers of k are $p\phi$, $\tfrac{1}{2}p(p-1)\phi$; therefore, &c.

To illustrate geometrically: let $U = 0$ be the equation of a conic, and ϕ its discriminant; then it will be seen that the evectant of ϕ is the contravariant Σ, where $\Sigma = 0$ is the tangential equation of the conic. Now, the substitution in question gives to U the form $(\alpha x + \beta y + \gamma z)^2$; hence the discriminant of ϕ vanishes, and the simultaneous vanishing of Σ expresses the fact that the line $\alpha x + \beta y + \gamma z = 0$ is a tangent to the conic

$$(\alpha x + \beta y + \gamma z)^2 = 0.$$

Another illustration may be taken from the cubic curve, the evectant of whose invariant S gives the Cayleyan; the invariant vanishes when the cubic becomes $(\alpha x+\beta y+\gamma z)^3 = 0$, and at the same time the tangential equation of the Cayleyan is satisfied by the coordinates of the line $\alpha x+\beta y+\gamma z = 0$.

8972. (W. J. C. Sharp, M.A.)—The Hessian of a cubic curve is the envelope of the straight lines two of the poles of which with respect to the cubic coincide.

Solution.

Instead of the cubic let us consider the general n-ic. To obtain all the poles of any line we may take any two of its points P, P′, and construct their first polars as to the n-ic; these curves will intersect in the $(n-1)^2$ poles of the line PP′. Now, let P, P′ be consecutive points on the Steinerian of the n-ic; then, by the fundamental properties of the Steinerian and Hessian, the first polar of P will have a double point Q on the Hessian, and the first polar of P′ will have a consecutive double point Q′; and, in the limit, each branch of one polar will intersect the corresponding branch of the consecutive polar on their envelope, while the non-corresponding branches will, in the limit, intersect at Q on the Hessian; hence Q will represent two of the $(n-1)^2$ intersections, or two coincident poles of the line PP′, which itself becomes a tangent to the Steinerian. Hence this curve is the envelope of the straight lines two of whose poles with respect to the n-ic coincide. The theorem in question is a particular case of this, for it is well known that in the case of the cubic the Steinerian and Hessian are identical. [A direct proof of this Question 8972, by the Proposer, will be found in Art. xi. of a paper on "Cubic Curves" in the *Quarterly Journal of Mathematics*, Vol. xvi., p. 300.]

9154. (Professor Haughton, F.R.S.)—If $2H/a^2$ be the quantity of sun-heat falling perpendicularly on an area equal to the section of the Earth at the mean distance a from the Sun in the unit of time, and if δ be the Sun's north declination; prove that the shares of heat received by the two hemispheres are

$$\text{Northern} = H(1+\sin\delta)/a^2, \quad \text{Southern} = H(1-\sin\delta)/a^2.$$

Solution.

Consider the hemisphere that is visible from the Sun; the projection of the visible semi-equator on the base of this hemisphere is a semi-ellipse, whose area is to that of the semi-equator as $\sin\delta$ is to unity; hence the areas of the visible portions of the northern and southern hemispheres will, as seen from the sun, be in the ratio of $1+\sin\delta$ to $1-\sin\delta$; and the required result is obtained by dividing the total quantity of heat $2H/a^2$ in this ratio.

10008. (D. BIDDLE.)—Prove that

$$\frac{1}{10^2}+\frac{1+3}{(10+5)^2}+\frac{1+3+4}{(10+5+6)^2}+\frac{1+3+4+5}{(10+5+6+7)^2}+\ldots=\frac{1}{4}.$$

Solution.

The nth term easily reduces to $2(n^2+3n-2)/(n^2+7n+12)^2$, which resolves into the partial fractions

$$4/(n+4)^2-4/(n+3)^2-2/(n+4)+2/(n+3);$$

therefore the sum to n terms is evidently

$$4/(n+4)^2-4/4^2-2/(n+4)+2/4,$$

which reduces to $n^2/4(n+4)^2$, and approaches the limit $1/4$ as n becomes infinite.

[The general method of summing series of this kind is explained by Mr. WOOLHOUSE in the foregoing Appendix I. to this Volume, pp. 148—152, and confirmation of the result in the particular case is incidentally given on p. 146 of the same Appendix.]

8141. (W. J. C. SHARP, M.A.)—If Δ be the discriminant of a quadric $U=0$; show that Δ is positive, negative, or zero, according as two distinct generators, no real generators, or two coincident generators can be drawn through each point on the surface.

8709. (W. J. C. SHARP, M.A.) — Prove that, according as the discriminant of a quadric is positive, zero, or negative, there can be drawn through any point on it two distinct generators, two identical, or two imaginary.

Solution.

When the discriminant vanishes the quadric is a cone, and hence the tangent plane at any point meets the surface in two real coincident lines, and these are identical generators through the point.

When the discriminant does not vanish, the equation of the quadric can be brought by linear transformation to the form

$$ax^2+by^2+cz^2+du^2=0,$$

in which the tetrahedron of reference is any self-conjugate tetrahedron; the discriminant is then the product $abcd$, and differs from its former value only by a multiplier which is an even power of the modulus of transformation; hence it is unaltered in sign. (See SALMON'S *Three Dim.*, Arts. 67, 141, 200; and *Mod. Alg.*, Art. 119 *et seq.*) Now, the four coefficients a, b, c, d, cannot all have the same sign; hence, (1) when the discriminant is positive two of the coefficients are positive and two negative, and (2) when the discriminant is negative one of the coefficients differs in sign from the other three. In these cases the equation can be written in the respective

forms (1) .... $L^2-M^2=N^2-P^2$,

(2) .... $L^2-M^2=N^2+P^2$.

In the first case: the line represented by the equations

$$L-M=k(N-P),\quad k(L+M)=N+P, \qquad \text{[Art. 109.}$$

lies wholly in the surface, and so does the line whose equations are

$$L - M = k(N+P),$$
$$k(L+M) = N-P, \qquad [\text{Art. } 109.$$

hence there are in this case two real and distinct generators through any point on the surface.

In the second case: it is evident, by changing the sign of P^2, that the generators are imaginary. Therefore, &c.

[In space of odd dimensions, the sign of the discriminant of a quadric is not altered by changing the signs of all the coefficients, and hence has a geometrical significance, which it has not in space of even dimensions.]

8949. (W. J. C. SHARP, M.A.)—The unicursal quartic

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0$$

may be derived from the conic $a\xi^2+b\eta^2+c\zeta^2+2f\eta\zeta+2g\zeta\xi+2h\xi\eta = 0$ by a double process of reciprocation.

Solution.

When an n-ic curve is unicursal, it has its maximum number of double points, $\frac{1}{2}(n-1)(n-2)$, and conversely. (SALMON's *Higher Plane Curves*, Art. 44.) Hence a unicursal quartic has three given points; and, if their connectors be taken as reference-lines, its equation must take the form (Art. 283)

$$ay^2z^2+bz^2x^2+cx^2y^2+2fx^2yz+2gxy^2z+2hxyz^2=0,$$

which when divided by $x^2y^2z^2$ may be written as in question.

It is now required to find a method of tracing the quartic, being given the conic, in whose equation the corresponding terms have the same six coefficients respectively.

Let $x=1/\xi$, $y=1/\eta$, $z=1/\zeta$; then, if the point $(\xi\eta\zeta)$ describe the given conic, it is clear that the point (xyz) will describe the given quartic. But it is evident that these two points are symmetrically situated as to each of the angle-bisectors of the triangle of reference; hence we have a simple geometrical construction for passing from one tracing-point to the other. Another way of expressing the relation between these two points is by means of two reciprocations, viz.,—take the polar line of $(\xi\eta\zeta)$ as to the cubic $xyz=0$ (its equation is $x/\xi+y/\eta+z/\zeta=0$, and the line is easily constructed, being the "axis of homology"), then take the pole of this line as to the conic $x^2+y^2+z^2=0$; the coordinates of this pole are $1/\xi$, $1/\eta$, $1/\zeta$, and it will therefore be the required tracing-point (xyz). We may state this result in other words as follows: The reciprocal of the given conic as to the cubic $xyz=0$ is the fourth-class curve whose tangential equation is

$$a/\alpha^2+b/\beta^2+c/\gamma^2+2f/\beta\gamma+2g/\gamma\alpha+2h/\alpha\beta=0,$$

while the reciprocal of the latter curve as to the conic $x^2+y^2+z^2=0$ is the fourth-degree curve whose trilinear equation is

$$a/x^2+b/y^2+c/z^2+2f/yz+2g/zx+2h/xy=0.$$

8304. (R. KNOWLES, B.A.) — The median anti-parallel from A meets the side BC of the triangle ABC in a point D; prove that (1) BD : DC = $c^2 : b^2$, and (2) the equations of the three median anti-parallels are $$cy - bz = 0, \quad bx - ay = 0, \quad cx - az = 0.$$

Solution.

It is convenient to prove the latter part of the question first. The median anti-parallel from A makes the same angles with the sides AB, AC as the median from A makes with the sides AC, AB respectively; but the equation of this median is $by = cz$, therefore the equation of its anti-parallel is $bz = cy$, therefore &c.

Again, it follows that the coordinates of D are proportional to 0, b, c;

therefore $$\text{BD} : \text{DC} = c/\sin \text{B} : b/\sin \text{C} = c^2 : b^2.$$

[The PROPOSER remarks that he obtained the result in this question by considering the median anti-parallel as bisecting chords of circles, according to the definition given by Mr. TUCKER in the *Educational Times*; but he adds that the proof in the accompanying solution is much shorter and more direct.]

9279. (F. MORLEY, M.A.)—A cubic has a cusp at O, OA being the cusp-tangent. PQR is any chord, and the tangent from P to the curve touches it at T. Prove that OQ, OR are harmonic to OA, OT.

Solution.

The cuspidal cubic belongs to the class of unicursal cubics, *i.e.*, in which the coordinates of any point on the curve can be expressed in terms of a single parameter (*Higher Plane Curves*, pp. 180, 188).

Let the equation of the cubic be $x^2z = y^3$, in which the line x is the cusp-tangent; then the coordinates of any point on the curve may be taken as 1, θ, θ^3, where θ is a varying parameter. Any line $ax + by + cz = 0$ will meet the curve in three points whose parameters are found from the equation $a + b\theta + c\theta^3 = 0$; but the sum of the roots of this equation is zero; hence, if three points lie on a line, the sum of their parameters is zero. Now, let the parameters of P, Q, R be θ_1, θ_2, θ_3; then $\theta_1 + \theta_2 + \theta_3 = 0$. The equations of OP, OQ, OR are $y = \theta_1 x$, $y = \theta_2 x$, $y = \theta_3 x$; and the latter may be written $y + (\theta_1 + \theta_2) x = 0$. Hence OQ and OR are harmonic

with the lines $$y + (\theta_1 + \theta_2) x + (y - \theta_2 x) = 0,$$

and $$y + (\theta_1 + \theta_2) x - (y - \theta_2 x) = 0, \qquad \text{[}\textit{Conics}\text{, p. 63.}$$

that is, with the lines $y + \frac{1}{2}\theta_1 x = 0$, and $x = 0$.

But, if the tangent from the point θ_1 touch the curve at T, the parameter of T must be $-\frac{1}{2}\theta_1$, and the equation of OT must be $y + \frac{1}{2}\theta_1 x = 0$; therefore OQ and OR are harmonic with the lines OA and OT.

It will be seen that, if T_1, T_2, T_3 are the points determined according to the original statement of the question, then OT_1 and OT_2 are harmonic with OA and OR, and so on.

8952. (W. J. C. Sharp, M.A.)—If a sextic equation can be reduced to a cubic, that is to say, to an equation having three pairs of roots equal in value and opposite in sign, the skew invariant will vanish.

Solution.

This invariant is of the order 15 in the coefficients of the sextic (Salmon's *Higher Algebra*, pp. 260, 270); hence its weight $= \frac{1}{2}.6.15 = 45$; therefore, in every term of the invariant, one of the coefficients must have an odd suffix; and it follows that the invariant vanishes when $a_1 = 0$, $a_3 = 0$, $a_5 = 0$, *i.e.*, when the terms that have odd exponents are absent from the given sextic equation; or, in other words, when it can be solved as a cubic for $x^2 : y^2$.

[For another property of the semi-invariant, see the Proposer's solution of Professor Sylvester's Question 4139, on p. 85 of Vol. xliii.]

8525. (G. S. Carr, M.A.)—If P, S are real points, coordinates xy, $x'y'$; and Q, R imaginary points, coordinates $(\alpha + i\alpha', \beta + i\beta')$ and $(\alpha - i\alpha', \beta - i\beta')$; show briefly that the *real* line which joins the imaginary points of intersection of the imaginary pairs of lines (PQ, SR), (PR, SQ) is identical with the line obtained by substituting unity for i in the imaginary coordinates, and drawing the five lines accordingly.

Solution.

I propose to give a general geometrical method of making such constructions; from which it will appear that the statement in question is erroneous.

When in the imaginary coordinates above we substitute unity for i, we obtain $(\alpha + \alpha', \beta + \beta')$, $(\alpha - \alpha', \beta - \beta')$, the coordinates of two real points Q_1, R_1, which may be called the *graph-pair* of the conjugate imaginary pair Q, R.

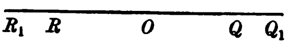
The following properties are easily proven, and are sufficient for the present purpose:—

1. *It is evident that the pairs* Q_1R_1 *and* QR *have the same mid-point, whose coordinates are* α, β.

2. *The line* QR *is identical with the real line* Q_1R_1; for these lines pass through the same point $(\alpha\beta)$, and make with the x-axis the same angle $\tan^{-1}\beta'/\alpha'$.

3. *If* O *be the mid-point, the segments* OQ, OQ_1, OR, OR_1 *are connected by the relations* $OQ = i.OQ_1$, $OR = i.OR_1$;

for $OQ_1^2 = OR_1^2 = \quad (\alpha'^2 + \beta'^2)$
and $OQ^2 = OR^2 = -(\alpha'^2 + \beta'^2)$, $\therefore$ &c.,



4. *The two point-pairs* QR, Q_1R_1 *are harmonic;* for, because

$$OQ_1 . OR_1 = -OQ_1^2 = OQ^2 = OR^2,$$

therefore the points Q_1, R_1 are inverse points with respect to the (imaginary) circle whose diameter is QR; therefore (by a well-known principle) Q_1, R_1 are harmonic as to Q, R.

5. *If* A *be any point on the line* QR, *and if* A′, A″ *be its respective harmonic conjugates as to the point-pair* QR, *and as to the point-pair* Q_1R_1, *then will* A′, A″ *be symmetrically situated with regard to* O.

For, because $\qquad OA . OA' = OR^2 = -OR_1^2$,

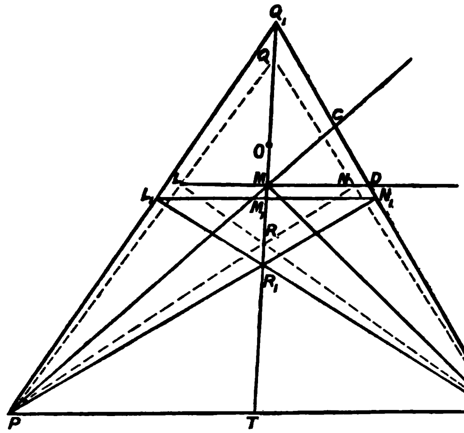
and $$OA \, . \, OA'' = OR_1^2,$$

therefore OA′ and OA″ differ only in sign.

These principles, taken in connection with the harmonic properties of a complete quadrilateral, give a simple real construction for the line required above.

Let Q, R be the conjugate imaginary points, and P, S the real points; then the dotted lines PQ, PR are conjugate imaginary lines, and so are the lines SQ, SR; therefore the coordinates of the intersections L, N differ only in the sign of i; therefore they are conjugate imaginary points, and the line LN is real.

Now let Q_1, R_1 be the graph-pair of Q, R, and complete the real quadrilateral as in the figure.



It is required to determine the line LN by a real construction, being given only the graph-pair Q_1R_1, and the real points P, S.

By the harmonic properties of the two complete quadrilaterals, the point M_1 is the harmonic conjugate of T as to Q_1, R_1, and the point M is the harmonic conjugate of T as to Q, R; therefore M and M_1 and are symmetric* as to the mid-point O (by 5 above).

Therefore the real point M is determined; and it is clear that the lines LMN, $L_1M_1N_1$ are *not* identical.

The line LMN is now easily constructed, for it is known that the pencil $M - Q_1CNS$ is harmonic; therefore, if D be taken harmonic conjugate of Q_1 as to C, S, the line LMND will be determined.

[Mr. Carr states that he would be grateful to Mr. MacMahon if he would point out any fallacy in the demonstration—presumably a rigid one—of the theorem in question which is given on page 549 of Mr. Carr's *Synopsis of Elementary Results in Pure and Applied Mathematics.*]

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* The points M, M_1 do not appear symmetric in the diagram, because the dotted lines (from which M is determined in the drawing) are merely ideal.

9001. (J. Brill, M.A.) — A solid is bounded by the surface $(x^2+y^2)/a^2 - z^2/b^2 = 1$ and the planes $z = \pm c$. Two smooth rods, which are joined at their upper extremities, make equal angles α with the vertical, the angle between the vertical planes drawn through the rods being β. Prove that, if the solid be placed on the rods with its axis of figure horizontal, it will be in equilibrium, provided that

$$b^2(b^2+k^2)\operatorname{cosec}^2 \tfrac{1}{2}\beta = a^2k^2\tan^2\alpha,$$

where $2k$ is the distance between the points of contact of the solid with the rods.

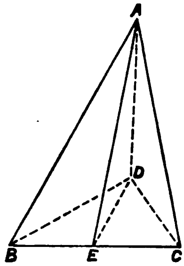
Solution.

In the position of equilibrium the points of contact lie in a vertical plane through the horizontal axis, and the normals to the surface at these points meet on the vertical axis; hence the rods lie in the tangent planes at these points, and intersect at a point on the line common to these planes.

Let the points of contact be $(x', +z')$, $(x', -z')$, the axis of x being vertically downwards and that of y horizontal; then the equations of the tangent planes are $\qquad xx'/a^2 \pm zz'/b^2 = 1,$

and these planes intersect in the line

$$x = a^2/x', \quad z = 0;$$



hence the x-coordinate of the point of junction of the rods is a^2/x'. In the figure let A be this point, and let B, C be the points of contact of the solid with the rods; let BE = EC, and let the line AD be vertical, and the lines BD, ED, CD horizontal; then $AD = x' - a^2/x'$, $EC = k$, $DAC = \alpha$, $EDC = \frac{1}{2}\beta$,

$$\tan\alpha = CD/AD, \quad \operatorname{cosec}\tfrac{1}{2}\beta = CD/EC,$$

therefore $\tan\alpha/\operatorname{cosec}\frac{1}{2}\beta = EC/AD = k/(x'-a^2/x')$;

but, since the points of contact lie on the given surface, we have $x'^2/a^2 - k^2/b^2 = 1$,

therefore $\qquad x'^2 - a^2 = a^2k^2/b^2,$

and $\qquad x' - a^2/x' = ak^2/b(b^2+k^2)^{\frac{1}{2}};$

therefore $\qquad \tan\alpha/\operatorname{cosec}\frac{1}{2}\beta = b(b^2+k^2)^{\frac{1}{2}}/ak$, therefore, &c.

8963. (W. J. C. Sharp, M.A. Suggested by Quest. 7545.)—The points on a right line have a 1 to 1 correspondence with the rays of a pencil in the same plane. Show that the lines drawn through the points parallel to their corresponding rays touch a parabola of which the line is a tangent.

Solution.

Take the right line for axis of x, and let the vertex of the pencil be at the point $(0, c)$; let any ray cross the x-axis at the point $(r, 0)$, and let

the point which corresponds to it be at $(r', 0)$; then, by hypothesis, r and r' are connected by a fixed relation of the form

$$rr' + lr + mr' + n = 0,$$

for this is the only form that will give a single value of r for a given value of r', and a single value of r' for a given value of r. Now the equation of the line through $(r', 0)$ parallel to the ray through $(r, 0)$ is

$$ry + c(x - r') = 0;$$

it is required to find the envelope of this line, when the parameters r, r' are subject to the above fixed relation. We have, by differentiating, and eliminating dr, dr', $\quad ry + r'c + lc + my = 0,$

therefore $\quad 2yr = -(cx + my + lc), \quad 2cr' = (cx - my - lc);$

hence, on eliminating r, r', we find, for the envelope, the parabola represented by $\quad (cx - my)^2 + 2c(lcx + lmy - 2ny) + l^2c^2 = 0,$

which may also be written in the form

$$c^2(x + l)^2 = y(2cmx - m^2y - 2clm + 4nc).$$

The first form gives the equations of a diameter, and of the tangent at its extremity; the second form gives the equations of a pair of tangents (of which the axis of x is one), and the equation of their chord of contact.

[From the first two equations we have

$$r^2y + r\{cx + cl + my\} + c(mx + n) = 0,$$

and the envelope is $\quad 4y(mx + n) = (cx + cl + my)^2,$

which easily reduces to the above results.]

10053. (W. J. C. Sharp, M.A. Extension of Question 9513.)—If common tangents be drawn to a curve of the mth class and to a curve of the second class, and if these be arranged in m pairs, and from their m intersections the other tangents be drawn to the first curve, prove that they will all touch a curve of class $m-2$. This theorem reduces to Question 9513, if the conic be replaced by two points, and if these points become coincident.

Solution.

This extension, mentioned in my solution of Question 9513, (Vol. L., p. 105), may be at once obtained from the following theorem, which is the reciprocal of the familiar theorem used in that solution. If of the m^2 common tangents of two m-class curves, mp touch a p-class curve, the remaining $m(m-p)$ common tangents will touch an $(m-p)$-class curve.

In the present case, the other m-class curve is made up of the "m intersections" mentioned in the question; but, of the entire m^2 tangents through these points to the given m-class curve, we have one set of $2m$ touching a 2-class curve, hence the remaining set of $m(m-2)$ will touch an $(m-2)$-class curve. A further extension is obvious.

8345. (Professor Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—If s be the arc of the cardioid $2r = a(1+\cos\phi)$, and σ the corresponding arc of the cycloidal curve described by the cusp when the cardioid rolls on a right line, show that $\left(\dfrac{ds}{d\phi}\right)^2 + \dfrac{a}{3}\cdot\dfrac{d\sigma}{d\phi} = a^2.$

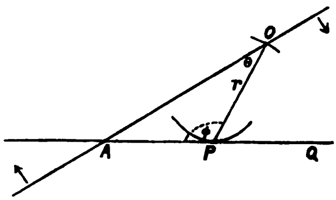
8906. (Professor Nilkantha Sarkar, M.A.)—If s be the arc of the equilateral hyperbola $r^2\cos 2\phi = a^2$, and σ the corresponding arc of the roulette described by the pole when the hyperbola rolls on a right line, prove that
$$\left(a\frac{d\phi}{ds}\right)^4 + \left(\frac{a}{2}\cdot\frac{d\phi}{d\sigma}\right)^4 = 1.$$

9121. (Professor Byomakesa Chakravarti, M.A.)—If s be the arc of the lemniscate $r^2 = a^2\cos 2\phi$, and σ the corresponding arc of the roulette described by the pole when the lemniscate rolls on a right line, show that
$$\left(a\frac{d\phi}{ds}\right)^4 + \left(\frac{2}{3a}\cdot\frac{d\sigma}{d\phi}\right)^4 = 1.$$

Solution.

That there is a mistake in the working out of each of these results will appear if we consider first the following question:—

Let the curve defined by the polar equation $r^n = a^n\cos n\theta$, roll on a fixed right line; required the "intrinsic equation" of the roulette described by the pole.



Let PQ be the fixed line, P the point of contact of the rolling curve with this line at any instant, O the pole which describes the roulette in question, OA the initial line from which θ is measured and which is fixed relatively to the rolling curve; let r, θ be the polar coordinates of P with reference to this pole and this initial line, ϕ the angle OPA which r makes with the tangent at its extremity; then, since P is the centre of instantaneous rotation, OP is normal to the roulette described by O, and in an infinitesimal time δt it is evident that the angle PAO is altered by the quantity $\delta\theta + \delta\phi$, which is therefore the angle turned through by the line OA, and also by the line PO in the same infinitesimal instant; therefore the arc $\delta\sigma$ described by O in the time δt is $r(d\theta + \delta\phi)$. But
$$\tan\phi = r\frac{d\theta}{dr} = -\cot n\theta,$$
therefore $$\phi = n\theta + \tfrac{1}{2}\pi, \quad\text{and}\quad \delta\phi = n\delta\theta,$$
therefore $$n\,d\sigma/d\phi = (n+1)\,r = (n+1)\,a(\cos n\theta)^{1/n};$$
but $$\cos n\theta = \sin\phi,$$
therefore $$\left[\frac{n}{(n+1)\,a}\cdot\frac{d\sigma}{d\phi}\right]^n = \sin\phi:$$
this is a relation between σ, the arc of the roulette, and ϕ, the angle which the normal makes with the fixed line; hence it may be considered the intrinsic equation of the roulette described by O.

In the next place, let us find the value of $ds/d\theta$ in the rolling curve. We have
$$\sin\phi = \frac{d\theta}{ds} = a(\cos n\theta)^{1/n} \cdot \frac{d\theta}{ds} = a(\sin\phi)^{1/n}\frac{d\theta}{ds},$$
therefore
$$a\frac{d\theta}{ds} = (\sin\phi)^{(n-1)/n},$$
therefore
$$\left(a\frac{d\theta}{ds}\right)^{n/(n-1)} = \sin\phi.$$

By equating the two expressions thus obtained for $\sin\phi$, we find the correct relation between ds and $d\sigma$. The proposers apparently mistook one of these *sines* for a *cosine*, and then equated the sum of the squares of the two expressions to unity; this would give
$$\left(a\frac{d\theta}{ds}\right)^{2n/(n-1)} + \left(\frac{n}{(n+1)\,a}\cdot\frac{d\sigma}{d\phi}\right) = 1,$$
which (except for the confusion of θ and ϕ) agrees with 8345 when $n = \frac{1}{2}$, with 8906 when $n = -2$, and with 9121 when $n = 2$.

8296. (W. J. C. Sharp, M.A.)—If a binary quantic be an exact pth power of another such quantic, prove that (1) its Hessian is a multiple of the lower quantic raised to the power of $2p-2$ and its Hessian; and (2) whenever a binary quantic measures its own Hessian, it is an exact power of another of a lower order.

Solution.

Take any binary quantic
$$u = a_0x^n + na_1x^{n-1}y + \tfrac{1}{2}n(n-1)\,a_2x^{n-2}y^2 + \dots$$
$$= a_0(x-\alpha_1 y)(x-\alpha_2 y)\dots(x-\alpha_n y).$$

Its Hessian $H(u)$ differs only by the factor $n^2(n-1)^2$ from the covariant
$$\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2,$$
and is of the degree $2(n-2)$ in x, y, and of the order 2 in the coefficients $a_0, a_1, \dots$.

Again, the covariant
$$\Sigma(\alpha_1-\alpha_2)^2(x-\alpha_3 y)^2\dots(x-\alpha_n y)^2$$
has the same degree and order as the Hessian; and the "source" of each of these covariants is (Salmon, p. 134) of weight $\frac{1}{2}[n\,.\,2-2(n-2)] = 2$; hence they have the *same* source $a_0a_2 - a_1^2$, and the expression above differs only from $H(u)$ by a numerical multiplier.

We determine this multiplier by taking the special form
$$u = a_0x^n + na_1x^{n-1}y,$$
and find
$$-n^2(n-1)\,H(u) = a_0^2\,\Sigma(\alpha_1-\alpha_2)^2(x-\alpha_3 y)^2\dots(x-\alpha_n y)^2,$$
which is a useful general expression for the Hessian of a binary quantic in terms of the roots.

Now, to prove (1), let $u = v^p$, where v is a binary quantic of the mth degree; then the roots of u may be arranged as follows:—

$$\begin{array}{l} \alpha_1 = \alpha_2 = \dots = \alpha_p, \quad \alpha_{p+1} = \alpha_{p+2} = \dots = \alpha_{2p}, \quad \alpha_{2p+1} = \alpha_{2p+2} = \dots = \alpha_{3p}, \\ \dots \qquad \dots \qquad \dots \qquad \dots \qquad \dots \\ \alpha_{(m-1)p+1} = \alpha_{(m-1)p+2} = \dots = \alpha_{mp}, \end{array}$$

and the expression on the right reduces to

$$\begin{aligned} pa_0^2 \Sigma (\alpha_p - \alpha_{2p})^2 (x-\alpha_p y)^{2(p-1)} (x-\alpha_{2p} y)^{2(p-1)} (x-\alpha_{3p} y)^{2p} \dots (x-\alpha_{mp} y)^{2p} \\ = p\left[(x-\alpha_p y)(x-\alpha_{2p} y)\right] \dots (x-\alpha_{mp} y)^{2(p-1)} \\ \times a_0^2 \Sigma (\alpha_p - \alpha_{2p})^2 (x-\alpha_{3p} y)^2 \dots (x-\alpha_{mp} y)^2, \\ = -pv^{2(p-1)} \times m^2 (m-1) \mathrm{H}(v); \end{aligned}$$

but $\qquad p = n/m,$

therefore $\qquad n(n-1)\mathrm{H}(u) = m(m-1) v^{2(p-1)} \mathrm{H}(v).$

To prove (2): Since, by the given condition, $\mathrm{H}(u)$ is divisible by (u), therefore $\quad (u^2)/(x-\alpha_1 y)^2 (x-\alpha_2 y)^2 + (u^2)/(x-\alpha_1 y)^2 (x-\alpha_3 y)^2 + \dots$ is divisible by (u), for every value of x, therefore (u) is divisible by each of the factors $(x-\alpha_1 y)^2$, $(x-\alpha_2 y)^2 \dots$, and must contain each of the factors $x-\alpha_1 y$, $x-\alpha_2 y \dots$ at least twice; hence (u) is an exact power of another quantic of lower order.

8953. (W. J. C. Sharp, M.A.)—All the inflexional tangents of a quartic curve touch two curves, of classes four and six respectively. If the quartic be unicursal, these curves are only of the second and third classes.

Solution.

The condition that the line $lx+my+nz=0$ may touch the general quartic curve is $S^3 = 27T^2$, wherein S, T involve l, m, n in the fourth and sixth degrees respectively (Salmon, Arts. 92, 250); and the form of this tangential equation shows that the 24 common tangents of the fourth class curve $S=0$ and the sixth class curve $T=0$, are the inflexional tangents of the quartic.

In the special case of a unicursal quartic, the coordinates of any point on the curve can be expressed in the forms

$$\begin{aligned} x &= a\ \lambda^4 + 4b\ \lambda^3\mu + 6c\ \lambda^2\mu^2 + 4d\ \lambda\mu^3 + e\ \mu^4, \\ y &= a'\ \lambda^4 + 4b'\ \lambda^3\mu + 6c'\ \lambda^2\mu^2 + 4d'\ \lambda\mu^3 + e'\ \mu^4, \\ z &= a''\lambda^4 + 4b''\lambda^3\mu + 6c''\lambda^2\mu^2 + 4d''\lambda\mu^3 + e''\mu^4, \end{aligned}$$

wherein $\lambda : \mu$ is a varying parameter; and the four points common to the line $lx+my+nz=0$ and to the quartic curve are then determined from the equation $\quad a_0\lambda^4 + 4a_1\lambda^3\mu + 6a_2\lambda^2\mu^2 + 4a_3\lambda\mu^3 + a_4\mu^4 = 0,$

wherein a_0, a_1, &c. are linear functions of l, m, n, a, b, c, &c.; hence the line will touch the quartic when this equation has equal roots, *i.e.*, when

$$(a_0a_4 - 4a_1a_3 + 3a_2^2)^3 = 27(a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3).$$

This is the tangential equation of the unicursal quartic, and shows that it has reduced to the sixth class, and that there are but six inflexional

tangents, which are the common tangents to the second class curve $a_0 a_4 - 4a_1 a_3 + 3a_2 = 0$, and to the third class curve

$$a_0 a_2 a_4 + 2a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3 = 0.$$

[The contravariant T is given for a particular form of the quartic in SALMON'S *Higher Plane Curves*, p. 259. $T = 0$ is the envelope of the lines which cut the quartic in four points which are harmonically conjugate.]

8622. (W. J. C. SHARP, M.A.)—Prove that (1) no two curves which are defined by linear relations between the focal distances from any number of common foci, except confocal conics, can cut at right angles; and (2) in particular two circular cubics or bicircular quartics which have a common focal circle and their foci common upon it cannot cut orthogonally.

Note.

This question seems to need a more accurate statement: for, take the curve defined by the equation

$$a_1 r_1 + a_2 r_2 + a_3 r_3 + a_4 r_4 = 0\,;$$

to find the direction of the tangent at any point, we have

$$a_1 \frac{dr_1}{ds} + a_2 \frac{dr_2}{ds} + a_3 \frac{dr_3}{ds} + a_4 \frac{dr_4}{ds} = 0,$$

i.e., $$a_1 \cos \alpha_1 + a_2 \cos \alpha_2 + a_3 \cos \alpha_3 + a_4 \cos \alpha_4 = 0,$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the angles between the tangent line and the focal radii; hence, if we take distances a_1, a_2, a_3, a_4 along the corresponding focal radii, and regard them as forces, the direction of their resultant will be normal to the curve. Now suppose we are given a point $(r_1' r_2' r_3' r_4')$ on the curve, at which the tangent makes the given angles $\alpha_1', \alpha_2', \alpha_3', \alpha_4'$ with the focal radii; and suppose it is required to determine another curve of the same family that shall pass through this point and cut the given curve orthogonally at the point. Then the coefficients of the new curve are to fulfil the conditions $$b_1 r_1' + b_2 r_2' + b_3 r_3' + b_4 r_4' = 0,$$

$$b_1 \sin \alpha_1' + b_2 \sin \alpha_2' + b_3 \sin \alpha_3' + b_4 \sin \alpha_4' = 0\,;$$

these conditions seem to be *sufficient* as well as necessary, hence I do not see why there are not an infinite number of curves, whose equations are of the form $$b_1 r_1 + b_2 r_2 + b_3 r_3 + b_4 r_4 = 0,$$

passing through the given point and cutting the given curve orthogonally.

[Mr. SHARP remarks that the statement in the Question is too absolute. It should read as follows:—" (1) Two curves which are defined by linear relations between the focal distances from any number of common foci, do not, in general, cut at right angles. Nor (2) in particular do two circular cubics or bicircular quartics which have a common focal circle and their foci common upon it." This is proved thus: If r, r', r'', &c. be the distances of a point from fixed points whose Cartesian coordinates are

$$(h, k),\ (h', k'),\ (h'', k''),\ \&c.,$$

now $$dr = \frac{x-h}{r}\,dx + \frac{y-k}{r}\,dy\,;$$

therefore, if the distances be connected by a linear relation (in which case the points are all foci of the curve represented in Question 7477), say

$$ar + br' + cr'' + \dots = 0,$$

$$\frac{dy}{dx} = -(a\cos\theta + b\cos\theta' + c\cos\theta'' + \dots) / (a\sin\theta + b\sin\theta' + c\sin\theta'' + \dots),$$

θ, θ', θ'', &c. being the angles made by the focal distances with the axis of x.

And if $$a'r + b'r' + c'r'' + \dots = 0$$
be another curve having the same foci, for this

$$\frac{dy}{dx} = -(a'\cos\theta + b'\cos\theta' + c'\cos\theta'' + \dots) / (a'\sin\theta + b'\sin\theta' + c'\sin\theta'' + \dots);$$

and therefore, if the curves cut orthogonally at (x, y),

$$(a\cos\theta + b\cos\theta' + c\cos\theta'' + \dots)(a'\cos\theta + b'\cos\theta' + c'\cos\theta'' + \dots)$$
$$+ (a\sin\theta + b\sin\theta' + c\sin\theta'' + \dots)(a'\sin\theta + b'\sin\theta' + c'\sin\theta'' + \&c.) = 0.$$

If $$ar + br' + c = 0, \quad a'r + b'r' + c' = 0,$$
it is easily seen this holds for all values of r and r' provided

$$\frac{a}{b} = -\frac{a'}{b'} = 1,$$

so that confocal conics always cut at right angles. If

$$ar + br' + cr'' = 0, \quad a'r + b'r' + c'r'' = 0,$$

this becomes

$$aa' + bb' + cc' + (ab' + a'b)(\cos\theta\cos\theta' + \sin\theta\sin\theta')$$
$$+ (bc' + b'c)(\cos\theta'\cos\theta'' + \sin\theta'\sin\theta'') + (ca' + c'a)(\cos\theta''\cos\theta + \sin\theta''\sin\theta) = 0,$$

which will not hold independently of the value of θ unless

$$aa' + bb' + cc' = 0, \quad ab' + a'b = 0, \quad bc' + b'c = 0, \quad ca' + c'a = 0,$$

which last three equations are inconsistent unless a, a'; b, b' or c, c' be zero, and so in the general case.]

8739. (Professor Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—The sun shines on a hill which stands in the form of a right circular cone on a plain. Show that the bounding lines of the shadow cast on the plain *always* touch the base of the hill.

Solution.

For the two tangent planes passing through a given point and touching a right circular cone, there is the following familiar construction: Let the line joining the vertex V to the given point S meet the plane of the base in A; from A draw two lines touching the circumference of the base in T, T′: these lines determine with the point S the two tangent planes.

Now, let the point S be supposed *luminous;* and (1) let A be outside the circumference and not at infinity; then, when S does not lie between A and V, the point A will be the shadow of V, and the two tangent lines AT, AT′ will be the shadows of the generators VT, VT′, and will be the bounding lines of the shadow of the cone; but, when S lies between A

and V, the bounding lines will be AT, AT′ *produced;* (2) when A is at infinity, the line VS is parallel to the base, and the bounding lines are tangents at the extremities of a diameter, and are each parallel to VS; (3) when A is not outside the circumference, the boundary of the shadow is the circumference of the base; therefore, &c.

7477. (W. J. C. Sharp, M.A.)—If a curve be defined by a linear relation among the distances of a point upon it from any number of fixed points, each of these is a focus.

8957. (W. J. C. Sharp, M.A.)—If a rational relation subsists between the distances of any point on a curve from two or more fixed points, and involves an odd power of one of them, the point from which this is measured is a focus, *i.e.*, the real intersection of two tangents to the curve from the circular points at infinity.

Solution.

(7477.) Let the linear relation be

$$a_1r_1+a_2r_2+\ldots+a_nr_n=0;$$

then, taking the first fixed point as origin, the Cartesian equation of the curve is

$$a_1(x^2+y^2)^{\frac{1}{2}}+a_2\left[(x-x_2)^2+(y-y_2)^2\right]^{\frac{1}{2}}+\ldots+a_n\left[(x-x_n)^2+(y-y_n)^2\right]^{\frac{1}{2}}=0.$$

This may be written for shortness in the form

$$u_1^{\frac{1}{2}}+u_2^{\frac{1}{2}}+u_3^{\frac{1}{2}}+\ldots u_n^{\frac{1}{2}}=0,$$

and when rationalised will be equivalent to

$$(u_1^{\frac{1}{2}}+u_2^{\frac{1}{2}}+u_3^{\frac{1}{2}}+\ldots)(u_1^{\frac{1}{2}}+u_2^{\frac{1}{2}}-u_3^{\frac{1}{2}}+\ldots)(u_1^{\frac{1}{2}}-u_2^{\frac{1}{2}}+u_3^{\frac{1}{2}}+\ldots)\ldots=0;$$

wherein the terms $u_2^{\frac{1}{2}}, u_3^{\frac{1}{2}}\ldots$ take all possible combinations of sign; hence the number of factors will be 2^{n-1}, and the rationalised equation may be written $$u_1^r+bu_1^{r-1}+cu_1^{r-2}+\ldots+su_1+t=0,$$

where $2r=2^{n-1}$, and the coefficients $b, c, \ldots$ are rational functions of $u_2, u_3\ldots$. Now the absolute term t (being the result of making $u_1=0$ in the product above) must be a complete square; hence, replacing u_1 by $a_1^2(x^2+y^2)$, u_2 by $a_2^2(x^2+y^2+l_2)$, u_3 by $a_3^2(x^2+y^2+l_3)$, &c., where $l_2, l_3\ldots$ are linear functions of x, y, the equation of the curve takes the form

$$\mathrm{U}(x^2+y^2)+\mathrm{V}^2=0,$$

in which U is of the degree $2(r-1)$ in x, y, and V of the degree r, where $r=2^{n-2}$; hence the locus represented by $x^2+y^2=0$, viz., the two imaginary lines through the origin and through the two circular points at infinity, must touch the curve at two of the points where $\mathrm{V}=0$ meets it. Therefore the origin (the first fixed point above) is the real intersection of two of the tangents to the curve from the circular points at infinity, and is thus, by definition, a focus. Similarly for each of the other fixed points.

It may be added that, if u_n be regarded as constant, the above investigation will include the case in which the given relation contains a constant term.

(8957.) In this generalisation of the last question, $u_1^{\frac{1}{2}}$ is replaced by $u_1^{\frac{1}{2}m}$, where m is odd, and u_2, u_3, ... are replaced by any rational functions of u_1, u_2, u_3 ... ; it will be seen that a similar argument will apply.

8406. (J. J. Walker, M.A., F.R.S.)—A focus (xyz) of the conic $u = 0$ (or $v = 0$ in $\xi\eta\zeta$) being determined by the conditions that

$$\left(\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}\right)^2-4vw=0$$

should be a circle, say $f(xyz) = \phi(xyz) = \psi(xyz)$; show that the square of the eccentricity for that focus is equal to $\chi(xyz) : \psi(xyz)$, where $\chi = 0$ is the condition that the directrix, viz. the chord of contact above, should pass through one of the circular points at infinity.

Solution.

Regarding ξ, η, ζ as current coordinates, the above equation represents the pair of tangents drawn from (xyz) to the conic $u = 0$ (Salmon, Art. 294), and the equation

$$\xi\frac{du}{dx}+\eta\frac{du}{dy}+\zeta\frac{du}{dz}=0$$

represents the polar line of the point (xyz).

When the former equation fulfils the conditions for representing a circle, it may be regarded either as the equation of the pair of lines joining (xyz) to the imaginary circular point at infinity, or as the equation of an infinitely small circle situated at (xyz); in either case, the locus has double contact with the conic, the chord of contact being the line whose equation is written above; and from either point of view, the point (xyz) is a *focus* of the conic [Arts. 258 (a), 261], and the chord of contact is the corresponding directrix. Now, let $(\xi'\eta'\zeta')$ be any point on the conic. In order to express its distance from the focus, let us first consider the general problem of finding the length of the tangent from a point $(\alpha'\beta'\gamma')$ to the circle $\quad a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0.$

Reverting for a moment to the Cartesian system, let the Cartesian equations of the three reference-lines be

$$x\cos\alpha+y\sin\alpha-p=0,\ x\cos\beta+y\sin\beta-p'=0,\ x\cos\gamma+y\sin\gamma-p''=0,$$

and let the Cartesian coordinates of the point $(\alpha'\beta'\gamma')$ be x', y'; then, substituting x', y' in the equation of the circle, and dividing by the coefficient of x^2, or of y^2, we find the square of the tangent

$$=\frac{a(x'\cos\alpha+y'\sin\alpha-p)^2+b(x'\cos\beta+y'\sin\beta-p')^2+\dots}{a\cos^2\alpha+b\cos^2\beta+c\cos^2\gamma-2f\cos\beta\cos\gamma-2g\cos\gamma\cos\alpha-2h\cos\alpha\cos\beta},$$

which (on replacing the coefficient of x^2 by half the sum of the coefficients of x^2 and of y^2, and returning to the trilinear system) becomes

$$\frac{a\alpha'^2+b\beta'^2+c\gamma'^2-2f\beta'\gamma'-2g\gamma'\alpha'-2h\alpha'\beta'}{\frac{1}{2}(a+b+c-2f\cos A-2g\cos B-2h\cos C)}.$$

Applying this general result to the point-circle above, we see that the

squared distance of $(\xi'\eta'\zeta')$ from the focus (xyz), is

$$\frac{\left(\xi'\frac{du}{dx}+\eta'\frac{du}{dy}+\zeta'\frac{du}{dz}\right)^2-v'u}{\frac{1}{4}(a'+b'+c'-2f'\cos A-2g'\cos B-2h'\cos C)},$$

wherein a', b', &c. are the coefficients of ξ^2, η^2, &c. in the equation of the point-circle, and are known functions of x, y, z; wherein also $v'=0$, since $(\xi'\eta'\zeta')$ is on the conic.

But the squared perpendicular distance of $(\xi'\eta'\zeta')$ from the directrix is

$$\frac{\left(\xi'\frac{du}{dx}+\eta'\frac{du}{dy}+\zeta'\frac{du}{dz}\right)^2}{\lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\lambda\nu\cos B-2\lambda\mu\cos C},$$

wherein $\qquad \lambda\equiv\frac{du}{dx},\quad \mu\equiv\frac{du}{dy},\quad \nu\equiv\frac{du}{dz}.$

Hence the squared eccentricity is

$$\lambda^2+\mu^2+\ldots/\tfrac{1}{4}(a'+b'+\ldots).$$

The numerator is the well-known expression whose vanishing is the condition that the line $\lambda x+\mu y+\nu z=0$ (in this case the directrix) should pass through one of the circular points at infinity.

I think the question does not describe the *denominator* with sufficient accuracy. It will be seen from the work above that it is *not* the common value of the three functions obtained by applying the general conditions given in SALMON (Art. 128).

Remarks by the PROPOSER.

The Question is stated with perfect accuracy, "the common value of the three functions obtained by applying the general conditions given in SALMON (Art. 129)," being the denominator as found by Mr. MCMAHON. The verification of this is only "easy" in the sense that "it may be shown with some difficulty"; but the work is not very long. Write the general conditions referred to, $p=q=r$; then it will be verified *without difficulty* that

$$(p-q)\cot C+(r-p)\cot B\equiv(b+c)\cos A-a\cos(B-C)$$
$$-2f+2h\cos B+2g\cos C,$$

which latter expression, therefore, and its two analogues vanish—as otherwise shown in the accompanying solution of an old question—if the general equation $\qquad (a\ldots h\between xyz)^2=0$

represents a circle. But in any triangle ABC,

$$\cos A\cos(B-C)=\sin^2 B+\sin^2 C-1,$$

so that, for the circle,

$$\Sigma\left\{a-a(\sin^2 B+\sin^2 C)+(b+c)\cos^2 A\right.$$
$$\left.-2f\cos A+2h\cos B\cos A+2g\cos C\cos A\right\}=0,$$

i.e. (in virtue of $\cos B\cos C=\sin B\sin C-\cos A\ldots$),

$3(\Sigma a-2\Sigma f\cos A)-2(p+q+r)=0$, or $p=q=r=\frac{1}{2}(\Sigma a-2\Sigma f\cos A)$.

Of course the proof of this common value was an essential part of the Solution of the Question, which has been hitherto unpublished.

The cited Question is the following one:—

6491. (J. J. Walker, F.R.S.)—Show that the expression

$$-4\,(\mathrm{A}\sin^2\mathrm{A} + \ldots + 2\mathrm{H}\sin\mathrm{A}\sin\mathrm{B})$$
$$+\,(a + b + c - 2f\cos\mathrm{A} - 2g\cos\mathrm{B} - 2h\cos\mathrm{C})^2$$

(where $\mathrm{A} = bc - f^2$, $\mathrm{H} = ab - ch$) may be thrown into the form of the sum of two squares; and point out the significance of the transformation."

Solution.

What is stated may be done in three ways; viz.,

$$\{(b-c)\sin\mathrm{A} + a\sin(\mathrm{B}-\mathrm{C}) + 2g\sin\mathrm{C} - 2h\sin\mathrm{B}\}^2$$
$$+\,\{(b+c)\cos\mathrm{A} - a\cos(\mathrm{B}-\mathrm{C}) - 2f + 2h\cos\mathrm{B} + 2g\cos\mathrm{C}\}^2,$$

and two similar forms, the verification of which is easy. Calling the above form $\mathrm{P}^2 + \mathrm{Q}^2$, $\mathrm{P}\sin\mathrm{A} = 0$ gives

$$b\sin^2\mathrm{A} + a\sin^2\mathrm{B} - 2h\sin\mathrm{A}\sin\mathrm{B} = a\sin^2\mathrm{C} + c\sin\mathrm{A} - 2g\sin\mathrm{C}\sin\mathrm{A},$$

one of the two well-known conditions that $(a \ldots h \between xyz)^2$ should represent a circle; and $(\mathrm{P}\cos\mathrm{B} + \mathrm{Q}\sin\mathrm{B})\sin\mathrm{C}$ gives a similar condition.

It is well known that the expression in the Question is the condition of equal axes in the ellipse, and its vanishing implies therefore two conditions, which are seen to be $\mathrm{P} = 0$, $\mathrm{Q} = 0$, leading to the better known conditions given in Salmon's *Conics*, Art. 128.

8648. (H. G. Dawson, M.A.)—If $u \equiv (abcfgh)(xyz)^2 = 0$ be a conic,

$$2\lambda = \frac{du}{dx},\quad 2\mu = \frac{du}{dy},\quad 2\nu = \frac{du}{dz},$$

$$\Omega^2 = \lambda^2 + \mu^2 + \nu^2 - 2\mu\nu\cos\mathrm{A} - 2\nu\lambda\cos\mathrm{B} - 2\lambda\mu\cos\mathrm{C},$$

$$\Theta = \begin{vmatrix} a & h & g & \sin\mathrm{A} \\ h & b & f & \sin\mathrm{B} \\ g & f & c & \sin\mathrm{C} \\ \sin\mathrm{A} & \sin\mathrm{B} & \sin\mathrm{C} & 0 \end{vmatrix},$$

prove that the diameter of the conic conjugate to the point xyz on it is $\Omega/\Theta^{\frac{1}{2}}$

8695. (H. G. Dawson, M.A.)—Deduce, as an instantaneous result of Question 8648, the equation of the director circle of the conic in Mr. Cathcart's form, *i.e.*, $\Theta' u - \Omega^2 = 0$ (Salmon's *Conic Sections*, Sixth Edition, p. 392).

Solution.

(86 48.) Take first the Cartesian equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0;$$

the equation of the tangent at $(x'y')$ may be written in either of the forms

$$x\ (ax'+hy'+g)+y\ (hx'+by'+f)+gx'+fy'+c=0,$$

$$x'\ (ax\ +hy\ +g)+y'\ (hx\ +by\ +f)+gx\ +fy\ +c=0.$$

Again, the coordinates α, β of the centre of the conic are determined from the equations $\quad a\alpha+h\beta+g=0,\quad h\alpha+b\beta+f=0$;

hence the length of the perpendicular from the centre on the tangent is

$$p=(g\alpha+f\beta+c)\,/\,\left[(ax'+hy'+g)^2+(hx'+by'+f)^2\right]^{\frac{1}{2}}$$

$$=\quad \frac{\Delta}{ab-h^2}\Big/\left[\quad ,,\qquad ,,\quad \right]^{\frac{1}{2}},$$

wherein $\qquad \Delta\equiv abc+2fgh-af^2-bg^2-ch^2.$

Now, we have for the length of the semi-diameter conjugate to that passing through $(x'y')$ the expression

$$b'=a_1a_2/p \qquad \text{[Salmon, p. 169]},$$

where a_1a_2 is the product of the semi-axes of the conic: to determine this product, we transform the equation to parallel coordinate axes through the centre; it takes the form

$$ax^2+2hxy+by^2=-\Delta\,/\,(ab-h^2),$$

which, when compared with the equation

$$\frac{x^2}{a_1^2}+\frac{y^2}{a_2^2}=1,$$

gives $\qquad \dfrac{1}{a_1^2}\cdot\dfrac{1}{a_2^2}=(ab-h^2)\Big/\dfrac{\Delta^2}{(ab-h^2)^2} \qquad$ [p. 157],

therefore $\qquad a_1a_2=\Delta\,/\,(ab-h^2)^{\frac{3}{2}},$

therefore $\qquad b'=\left[(ax'+hy'+g)^2+(hx'+by'+f)^2\right]^{\frac{1}{2}}/\,(ab-h^2)^{\frac{1}{2}}.$

The equivalent, in trilinear coordinates, of this result is at once obtained as in Salmon, p. 351; this numerator is the denominator of the fraction that gives the length of the perpendicular from any point to the tangent at $(x'y'z')$; and the vanishing of this numerator is the condition that the tangent at $(x'y'z')$ should pass through one of the circular points at infinity: hence it is a *covariant* of the system composed of the conic and these circular points. Again, $ab-h^2$ is the expression whose vanishing is the condition that the conic may be a parabola; it is an *invariant* of the system just mentioned, being the condition that the line joining the circular points should touch the conic. The general expression for this covariant

is (p. 352) $\qquad \lambda^2+\mu^2+\nu^2-2\mu\nu\cos A-2\nu\lambda\cos B-2\lambda\mu\cos C,$

wherein $\quad \lambda=ax+hy+gz,\quad \mu=hx+by+fz,\quad \nu=gx+fy+cz$;

and the general expression for the invariant is the determinant given on p. 267, remembering that the equation of the line at infinity is

$$x\sin A+y\sin B+z\sin C=0; \qquad \text{therefore } b'=\Omega/\Theta^{\frac{1}{2}}.$$

(8695.) I do not at present see any direct connection between the two questions. It is, however, easy to verify Mr. CATHCART's result; for the Cartesian equation of the director-circle given on p. 352, viz.,

$$C(x^2+y^2)-2Gx-2Fy+A+B=0,$$

is at once identified with the equation

$$(a+b)(ax^2+2hxy+by^2+2gx+2fy+c)=(ax+hy+g)^2+(hx+by+f)^2.$$

Now, in order to see how this equation is to be generalised, let us consider an "*a priori* reason" for its form, and at the same time show the the invariant (or covariant) character of each of its parts.

The equation $$(ax+hy+g)^2+(hx+by+f)^2=0$$

represents a pair of imaginary lines passing through the centre of the conic; at one of the four intersections let a tangent be drawn to the conic. It is clear from the discussion above that this tangent satisfies the condition for passing through one of the circular points at infinity; hence (by a well-known property of such a line) it may be regarded as the common limiting position of two consecutive imaginary tangents at right angles to each other, therefore the intersection in the question must be common to the conic and director-circle, and similarly for the other three intersections: this accounts for the right-hand side of the equation, and shows that it is to be generalised into the covariant Ω^2 of the last question.

To account for the factor $a+b$, we see that when $a+b=0$ the equation reduces to $$(ax+hy+g)^2+(hx+by+f)^2=0,$$

which can now be put in the form

$$x^2+y^2+2lx+2my+n=0,$$

showing that the pair of imaginary lines now pass through the circular points at infinity, and hence their equation may be regarded as that of a point-circle concentric with the conic; but when $a+b=0$ the conic is an equilateral hyperbola: hence the equation expresses the familiar fact that the director-circle reduces in this case to a point-circle. Thus $a+b$ is to be generalised into the invariant Θ'; and the equation of the director-circle is $$\Theta' u-\Omega^2=0.$$

[Mr. DAWSON remarks that, as he preserved no note of his original solution of this Question, he is unable to recall to memory its connexion with (8648). The following direct solution is, however, he thinks, interesting:—

"Let p, ϖ be respectively the perpendicular from any point of the director circle on its polar with regard to the ellipse, and the perpendicular from the centre on the same line; then, as easily seen,

$$-p\varpi=a^2b^2/(a^2+b^2).$$

Let x_0, y_0, z_0 be the coordinates of the centre, then

$$p=u/\Omega \quad \text{and} \quad \varpi=u_0/\Omega,$$

also $$1/a^2+1/b^2=-\Theta\Theta'/M^2\Delta,$$

and $$ax_0+hy_0+gz_0=M\Delta\sin A/\Theta,$$

with similar expressions for

$$hx_0 + by_0 + fz_0 \quad \text{and} \quad gx_0 + fy_0 + cz_0.$$

Hence $\quad u_0 = \mathrm{M}\Delta\,(x_0 \sin \mathrm{A} + y_0 \sin \mathrm{B} + z_0 \sin \mathrm{C})/\Theta = \mathrm{M}^2\Delta/\Theta,$

and the equation of the director circle is

$$-\Theta\Theta'/\mathrm{M}^2\Delta = -\Omega^2\Theta/\mathrm{M}^2\Delta \quad \text{or} \quad \Theta' u - \Omega^2 = 0.\text{"}]$$

[Mr. MACMAHON remarks that time will not permit him now to complete, for the present Volume, the Solutions of the Questions that follow the foregoing Question 8739, but that he proposes to finish them, along with others, in an Appendix to the next Volume.]

7418. (W. J. C. SHARP, M.A. Suggested by Quest 7227.)—Find in how many ways p sets of n things, the individuals of each of which are marked $1 . 2 \dots n$, can be permuted, so that no two individuals marked with the same number shall occupy the same position in any two sets.

Note.

See Vol. XL., p. 22.

[This question, which is in fact that of the game of treize or rencontre, is several times noticed in TODHUNTER's *History of the Theory of Probability*, where solutions are given by BERNOULLI, DE MOIVRE, and LAMBERT, of which the first is very similar to the Proposer's referred to above. The principal notices, where references to the original works are given, are pp. 93, 115, 153, 157, 239, 336, 534.]

8033. (W. J. C. SHARP, M.A. Suggested by Quest. 7536.)—If $3n-1$ points be given on a plane cubic and an $(n-\nu)$-ic curve be described through any $3n-3\nu-1$ of these, and any $(\nu+1)$-ic curve through the remaining intersection of this with the cubic and the other 3ν given points; prove (1) that this will cut the cubic in two additional points, the line joining which passes through the single point residual to the $3n-1$ given points; (2) enunciate the reciprocal proposition.

Note.

See Vol. XXXIV., p. 34; Vol. XLI., p. 69; Vol. XLVII., p. 137.

[Mr. SHARP remarks that this Question was intended to call attention to the application of Professor SYLVESTER's valuable theory of Residuation to Class Cubics. The fundamental property, the polar reciprocal of Question 2391 (*Reprint*, Vol. XLVII., p. 137), is,—"Let μ tangents to a class cubic be given; draw any curve of class ν to touch these, the remaining $3\nu-\mu$ (say μ') common tangents to the class cubic and class ν-ic may be called a first residuum to the given ones. Again, draw a class ν'-ic to

touch these μ' tangents to the class cubic, the remaining $3\nu'-\mu'$ common tangents to the class ν'-ic and the class cubic may be called a residuum of the second order, and so on. Whenever such a residuum consists of a single tangent to the class cubic, it is called a residual of the original μ tangents. Such a residual is dependent solely on the original μ tangents to the class cubic, and independent of the steps by which it is obtained."]

8542. (J. BRILL, M.A.)—PQR is a triangle circumscribed to a parabola. QR, RP, PQ touch the parabola at P′, Q′, R′, and meet another tangent in X, Y, Z. Prove that, if O be the point of concurrence of PP′, QQ′, and RR′,

$$\frac{PP'.YZ}{\sin QOR} = \frac{QQ'.ZX}{\sin ROP} = \frac{RR'.XY}{\sin POQ}.$$

Note.

The question implies that the segment of a variable tangent, intercepted by three fixed tangents to a parabola, have a constant ratio, which is easily proven by taking one position of the variable tangent as axis of y and the corresponding diameter as axis of x, when the three intercepts will be found to be

$$\tfrac{1}{2}(y_2-y_3),\ \tfrac{1}{2}(y_3-y_1),\ \tfrac{1}{2}(y_1-y_2),$$

whose mutual ratios are evidently independent of the particular diameter chosen for axis of x.

8580. (ARTEMAS MARTIN, LL.D.)—A given right cone, of specific gravity ρ, floats in water; find the inclination of its axis to the horizon.

Note.

The answer to this question is included in a Solution to Question 5190, by Mr. J. J. WALKER, Vol. XXVIII., p. 77; see also a question by the late Professor TOWNSEND, Vol. XXVII., p. 57.

8618. (J. BRILL, M.A.)—ABC is a triangle, and P and Q are two points within it; prove that

$$PA.QA.BC\cos(PAB-QAC)+PB.QB.CA\cos(PBC-QBA)$$
$$+PC.QC.AB\cos(PCA-QCB)=BC.CA.AB.$$

Note.

It is worthy of remark that this expression is independent of the position of the points P, Q. Compare Vol. XLII., p. 113; Vol. XLIX., p. 62.

8682. (Professor MATHEWS, M.A.) — Defining, with STANDT, an imaginary point by means of an involution without double points, and an imaginary plane by a similar involution of planes with a common axis; prove that three points will in general determine a plane, and show how to construct this plane geometrically when the points are (i.) two real and one imaginary, (ii.) two imaginary and one real, (iii.) all imaginary. (An imaginary plane contains an imaginary point when the involution which represents the former is in perspective with the involution defining the latter.)

Note.

See the solution to Mr. CARR's Question 8525, on p. 165 of this Appendix. The conjugate imaginary points Q, R are evidently the double points of the (negative) involution whose centre is O, and of which Q_1, R_1 are a real pair of conjugate elements; but in this case the imaginary points were first defined *algebraically*. STANDT, avoiding any reference to algebra, places the existence of imaginary points on a purely geometric foundation, by taking a negative involution as his definition of an imaginary point. In the same way the conjugate imaginary lines

$$(\alpha \pm i\alpha')\,x + (\beta \pm i\beta')\,y + (\gamma \pm i\gamma') = 0$$

are the double lines of a negative involution of which the lines

$$(\alpha \pm \alpha')\,x + (\beta \pm \beta')\,y + (\gamma \pm \gamma') = 0$$

are the conjugate rays that have the same angle-bisectors as the double lines; and this involution may be used to define these imaginary double lines.

Similarly for conjugate imaginary *planes*. When two rows of points, each in negative involution, are given, there is a known construction for determining the involution-pencil that is in perspective with each of the rows, or whose double lines pass through the double points of these rows; there are two solutions, corresponding to the two ways of pairing the points, so that two non-conjugate imaginary points determine an imaginary line.

This construction can be probably extended to the case asked for in question, viz.: Given three involution-rows in space, construct the involution of coaxial planes whose "double planes" pass through the double points of the rows.

8950. (W. J. C. SHARP, M.A.)—The p-ary n-ic has generally only

$$\frac{(n+1)\,(n+2)\ldots(n+p-1)}{1\,.\,2\ldots(p-1)} - p^2 + 1$$

independent invariants, p independent covariants, including the quantic itself, and p independent contravariants.

Note.

See SALMON, Arts. 121, 127, 145, 191, 192. (May not a *binary* quantic have more than *one* independent covariant besides the quantic itself?)

[Mr. SHARP remarks that the p-ary n-ic $(a, b, \ldots)(x, y \ldots)^n$ contains

$$\frac{(n+1)(n+2)\ldots(n+p-1)}{1.2\ldots p-1} \text{ terms,}$$

and therefore, if it be transformed into $(A, B, \ldots)(X, Y, \ldots)^n$ by the linear transformation (which involves p equations)

$$\left.\begin{array}{l} x = \lambda_1 X + \mu_1 Y + \ldots \\ y = \lambda_2 X + {}_2 Y + \ldots \\ \ldots \; \ldots \; \ldots \; \ldots \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(A)},$$

there will result $\frac{(n+1)(n+2)\ldots(n+p-1)}{1.2\ldots(p-1)}$ equations, giving the values of A, B, &c., between which and

$$\Delta = \begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \ldots \\ \lambda_2, & \mu_2, & \nu_2 \ldots \\ \lambda_3, & \mu_3, & \nu_3 \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

the constants of transformation $\lambda_1, \lambda_2 \ldots \mu_1, \mu_2 \ldots$ &c. may be eliminated, giving the number of resultants stated in the Question. If the p equations (A) be also used, p more resultants involving the variables (covariants), one of which is the quantic itself, will be obtained; and if $(\xi, \eta, \zeta \ldots)$ transforming into $(\Xi, \mathrm{H}, \mathrm{Z} \ldots)$ be contragredient variables, the p equations $\Xi = \lambda_1\xi + \lambda_2\eta + \lambda_3\zeta$, &c. will furnish another set of p contravariants. It is easily shown that a binary quantic can only have one independent covariant; for, let $f(x, y) \equiv \mathrm{U}$ be the quantic, and $\phi(x, y) \equiv \mathrm{V}$, $\chi(x, y) \equiv \mathrm{W}$ any two covariants, a relation between U, V, and W may always be obtained by eliminating x and y, and V and W are not therefore independent. A similar proof will show that a p-ary quantic cannot have more than $p-1$ independent covariants.]

8982. (Professor NEUBERG.)—Une droite se meut dans l'espace de manière que ses distances à deux points fixes A et B sont dans un rapport constant. Démontrer: 1°, que cette droite enveloppe une conique, lorsqu'elle se déplace dans un plan donné; 2°, qu'elle engendre un cône du second ordre, lorsqu'elle tourne autour d'un point fixe. Réciproquement, étant donnés une conique ou un cône du second ordre, trouver deux points A et B tels qu'il existe un rapport constant entre leurs distances à une tangente quelconque de la conique ou à une génératrice quelconque du cône.

Note.

(1) Let the moving line be $lx + my + n = 0$ in the plane of (xy), and let the fixed points be $(x_1, 0, z_1)$, $(-x_1, 0, z_2)$; then, from the given conditions, we have

$$z_1^2 + \frac{(lx_1+n)^2}{l^2+m^2} = k^2\left[z_2^2 + \frac{(lx_1-n)^2}{l^2+m^2}\right].$$

Hence the tangential equation of the envelope is of the form

$$Al^2 + Bm^2 + Cn^2 + 2Gln = 0,$$

and its Cartesian equation of the form

$$ax^2 + by^2 + 2gx + c = 0; \text{ therefore \&c.}$$

(2) Taking the fixed point as origin, let α, β, γ be the direction-angles of the moving line in any position; then the squared distance of $(x_1 y_1 z_1)$ from this line is

$$(y_1 \cos\gamma - z_1 \cos\beta)^2 + (z_1 \cos\alpha - x_1 \cos\gamma)^2 + (x_1 \cos\beta - y_1 \cos\alpha)^2.$$

Hence the given condition leads to a homogeneous relation of the second degree between $\cos\alpha$, $\cos\beta$, $\cos\gamma$; therefore &c.

The "réciproquement" may be left for further consideration, and is probably not difficult.

9144. (Captain H. BROCARD.)—L'équation de la glissette d'un point d'une courbe étant $x/y = f(y)$, l'équation différentielle de la roulette de ce point sera $dy/dx = -f(y)$, et réciproquement. Application à quelques exemples simples, point d'une circonférence, foyer d'une parabole, pôle d'une spirale logarithmique.

Note.

This question seems to need a more definite statement. (Compare Vol. XXXVII., p. 44.)

APPENDIX III.

UNSOLVED QUESTIONS.

2665. (By Prof. Crofton, F.R.S.) — 1. Given any closed convex boundary, without salient points: if dS be any element of the external area; T, T′ the tangents drawn from dS to the boundary; θ the angle they contain; and if the integration extend over the whole external area;

then prove that $$\iint \frac{\sin\theta}{TT'}\,dS = 2\pi^2.$$

2. If the integration extend over the annular space between the proposed boundary and an outer line along which θ has a constant value (α), the value of the integral is $2\pi(\pi-\alpha)$.

3. If the given boundary has salient points, then for every such point where the bounding line changes direction abruptly through an angle A, we must subtract $\frac{1}{2}A^2$ from the above values: then the value is $2\pi^2 - \frac{1}{2}\Sigma A^2$. For instance, for a regular polygon of n sides, the value will be

$$2\pi^2(1-1/n).$$

2707. (By the late M. Collins, B.A.)—Find what differential expressions or functions, on the usual supposition of dx being taken constant, are equivalent to the values of $\dfrac{dx\,.\,d^3y}{d^2y^2}$, $\dfrac{dx^2\,.\,d^4y}{d^2y^3}$, $\dfrac{dx^2\,.\,d^3y^2}{d^2y^4}$, taken or found on the supposition of dx^2+dy^2 being constant, y being any known or unknown function of x.

2708. (By Professor Crofton, F.R.S.)—Two equal spheres in contact are placed within a closed surface; determine the form of the envelope when its total surface is the least possible.

2746. (By J. J. Walker, F.R.S.)—Professor Rawlinson ("Ancient Monarchies") writes:—"The symbol of this king was the Crescent Moon *with the horns horizontal*, an appearance seldom observed in nature." It is proposed to discuss the problem in Spherical Astronomy which is suggested by this remark.

2753. (By Professor Wolstenholme, Sc.D.) — If a straight line be divided at random into four parts, (1) the chance that one of the parts shall be greater than half the line is $\frac{1}{2}$; also the respective chances that (2) three times, and (3) four times, the sum of the squares on the parts, shall be less than the square on the whole line, are $\frac{1}{18}\pi\sqrt{3}$ and $\frac{3}{500}\pi^2\sqrt{5}$.

2764. (By Samuel Roberts, M.A.)—Show that, if (a, a_1), (b, b_1) are points of contact of tangents from two points on a cubic curve, and (a, b), (a_1, b_1) have the same connective, then the four points lie on a conic which passes through their tangentials.

2760. (By Professor Crofton, F.R.S.)—1. Given two circles, whose radii are R and r, one wholly within the other; let a circle whose radius exceeds R be thrown at random on the plane: if it meets the outer circle, show that the chance of its also meeting the inner is $p = r / \mathrm{R}$.

2. Given any two convex figures, whose perimeters are l and L, one wholly within the other; let any moveable convex figure, which cannot be made to cut either of the others in more than two points, and which cannot lie wholly within the outer one, be thrown at random on the plane: if it meet the outer one, the chance of its also meeting the inner one is $p = l / \mathrm{L}$.

[In connexion with this problem, the following curious question occurs: —What are the conditions that a circle of given radius is incapable of cutting a given closed convex oval in more than two points?]

2796. (By R. Tucker, M.A.)—Find some other curves, besides those ormerly obtained, whose evolutes are similar to themselves.

2809. (By J. J. Walker, F.R.S.)—Given three rational and integral functions of x, viz., $f(x)$, $\phi(x)$, $\psi(x)$, it is proposed to find a criterion for the number of simultaneous positive values of the last two when the same root (all may be supposed real) of $f(x)$ is substituted in them for x.

2814. (By the late M. Collins, B.A.)—Find whether the common difference of three rational square numbers in Arithmetical Progression can ever be equal to 17.

2825. (By Professor Crofton, F.R.S.)—Given any two convex areas, draw the two common cross tangents AOA′, BOB′; then, if θ be the inclination of the tangents from any point P to the arcs AB, A′B′, θ denoting the exterior angle of the tangents drawn to the same arcs for points within the spaces AOB, A′OB′; prove that

$$\iint (\theta - \sin\theta)\, d\mathrm{S} = \tfrac{1}{2}\,(\mathrm{AA}' + \mathrm{BB}' - \text{arc AB} - \text{arc A}'\mathrm{B}')^2,$$

the integration extending over the space within the angles AOB′, A′OB, and the spaces AOB, A′OB′.

2826. (By Professor Burnside, M.A.)—When can the condition that three conics have a common point be expressed as $\mathrm{P}^{\frac{1}{2}} + \mathrm{Q}^{\frac{1}{2}} + \mathrm{R}^{\frac{1}{2}} = 0$?

2831. (By Professor Crofton, F.R.S.)—If R, S, T are three foci of a bicircular oval, and ρ', σ', τ' the tri-polar vectors of a point of contact of one of its double tangents from the centre of the circle RST; show that the equation of the oval is

$$\rho\rho'\,(\sigma'^2 - \tau'^2) + \sigma\sigma'\,(\tau'^2 - \rho'^2) + \tau\tau'\,(\rho'^2 - \sigma'^2) = 0.$$

2857. (By Professor Wolstenholme, Sc.D.)—Investigate the nature of the circular points at infinity, on the epicycloids and hypocycloids (which all pass through them); proving that the cardioid and the four-cusped hypocycloid have cusps at those points.

2860. (By T. Swainson.)—Being desirous of making a plan of one of the fields of my estate, which is in the form of an irregular pentagon, I measured its sides and found them to be as follows: AB = 6, BC = 4,

CD = 3, DE = 2, and EA = 1 chain, respectively. I also observed that the angles at the points B, C, and D were equal angles, and that the sun ranged with the angular points B and D at 9 h. 10 m. a.m. on the 10th day of February, 1868. I desire to know how to plot this field in its true position with regard to the true meridian from the above data.

2868. (By Professor Crofton, F.R.S.)—1. A bi-circular quartic passes through three fixed points, and two of its foci are fixed; find the locus of its two remaining foci, and show that it always passes through a fourth fixed point.

2. Show that, for either oval of a bi-circular quartic, two of the foci are internal and two external. Hence, when the two conjugate ovals composing the quartic are external to each other, each encloses two of the four foci.

2871. (By the Editor.)—An oblate spheroid, capable of turning on its centre, is placed at a fixed distance from a centre of force varying inversely as the square of the distance; find the oscillations of the spheroid.

2874. (By the late T. Cotterill, M.A.)—Seven points on a cubic locus have an opposite point on the curve: *i.e.*, a variable cubic through seven given points cuts a fixed cubic through the same points, in two other points collinear with a point on the fixed cubic. Construct for the opposite point when the fixed cubic breaks up into a conic through five points and a line through two.

2879. (By J. J. Walker, F.R.S.)—(1) Show that the six values of the Anharmonic Ratio of a Steiner's triad of points on an ellipse and their fourth are given by the equation

$$16x^2y^2\left\{\lambda(\lambda-1)+1\right\}^2-27a^2b^2\lambda^2(\lambda-1)^2=0,$$

where (x, y) are the coordinates of the fourth point, through which the osculating circles at the other three pass, referred to the semi-axes ab; and (2) determine the point (x, y) so that the four points may form a harmonic set.

2881. (By Artemas Martin, LL.D.)—A heavy stone was lying b feet from the centre of a circular race-ground whose diameter is $2a$ feet. A rope $(a-b)$ feet long was fastened to the stone, and a team was hitched to the other end of the rope, and driven once round the ring. Required the nature of the curve described by the stone on the ground, the distance it moved, and its distance from the outside of the ring.

2885. (By Artemas Martin, LL.D.)—Three horses are tethered in an equilateral triangular lot, each being fastened to a corner of the lot by a rope r feet long, and there are a square feet in the centre of the lot which is common to all three of the horses. Required the dimensions of the lot.

2896. (By J. Mason.)—There is a pipe $2\frac{1}{4}$ inches in diameter that conveys the water to the depth of 31 fathoms at East Castle Colliery; the top of the pipe is covered with 3 feet of water, and the descent is perpendicular. Required the number of gallons it will discharge per minute.

2912. (By N'Importe.)—Find three rational numbers such that, if their squares be diminished by 2, 3, 5 respectively, the three remainders shall be rational squares.

2925. (By J. J. Walker, F.R.S.)—Prove that the maximum value of the angle of intersection with an ellipse at D, of the circle passing through three points on the ellipse, the osculating circles at which meet in

D, is $\sin^{-1}\dfrac{a^2-b^2}{2(a^2+b^2)}$; and that in this case the centre of the circle lies on the chord common to the ellipse and its osculating circle at D.

2933. (By Artemas Martin, LL.D.)—A boy walked across a horizontal turn-table while it was in motion at a uniform rate of speed, keeping all the time in the same vertical plane. The boy's velocity is supposed to be uniform with respect to his track on the table, and equal to m times the velocity of a point in the circumference of the table. Required the nature of the *curve* he described on the table, and the distance he walked while crossing it:

1. When the motion of the table is *towards* him, (a) when $m>1$, (b) whem $m=1$, and (c) whem $m<1$.
2. When the motion of the table is *from* him, (a) when $m>1$, (b) when $m=1$, and (c) when $m<1$.

2971. (By Artemas Martin, LL.D.)—Show that the solution of the famous "Curve of Pursuit Problem," when the object pursued moves in the circumference of a circle and the pursuer starts from the centre, can be made to depend upon the solutions of the differential equations

$$d\theta = -\frac{dt}{n-\cos\phi}\ \ldots\ldots\ldots\ldots(1),\qquad tdt = r\left\{d\,(t\sin\phi)-ntd\phi\right\}\ldots\ldots\ldots(2);$$

where r is the radius of the circle, t the distance the two objects are apart at any time during the motion, ϕ the angle t makes with a tangent to the circle, and θ the arc described by the pursued object from the commencement of the motion, supposing the pursuer to move n times as fast as the pursued.

2976. (By Artemas Martin, LL.D.)—A chord is drawn through a given point in the surface of a given circle. Find (1) the average length of the chord, and (2) the average area of the segment cut off by the chord.

2984. (By Professor Genese, M.A.)—Deduce the following general principles:—(1) If the conditions for the construction of a series of straight lines be such that from every point of a given straight line *two* of the series may be drawn (*which are real when drawn through the point at infinity on the line*), and *if two and not more than two* of the series are in any the same direction, then the envelope of the series is a central conic.—(2) If through every point of the given straight line only *one* of the series can be drawn, the other conditions being as above, then the envelope is a central conic *touching* the given straight line.

3026. (By Professor Burnside, M.A.)—We cannot determine whether two conics intersect in real points from knowing the signs of any functions of their invariants; but it is possible to determine the nature of the intersections of a conic and cubic curve from knowing the signs of certain invariantal functions.

3027. (By the late T. Cotterill, M.A.)—Prove this theorem:—If a conic touch the sides of a quadrilinear and trilinear, an infinite number of point cubics pass through nine vertices.

1. Hence, if a conic touch the eight sides of two quadrilinears, their twelve vertices lie on a cubic. (Chasles states, that if five points and a line are given in space, one twisted cubic through the points will cut the line in two points.)
2. Find the two points by means of the theorem.

3105. (By Professor Cayley, F.R.S.)—**The following singular problem of literal Partitions arises out of the geometrical theory given in Professor** Cremona's Memoir, "Sulle Trasformazioni Geometriche delle Figure Piane," Mem. di Bologna, tom. V. (1865). It is best explained by an example:—A number is made up in any manner with the parts 2, 5, 8, 11, &c., viz., the parts are always the positive integers $\equiv 2 \pmod{3}$; for instance, $27 = 1 \,.\, 11 + 8 \,.\, 2$. Forming, then, the product of 27 factors $a^{11}(bcdefghi)^2$, this may be partitioned on the same type $1 \,.\, 11 + 8 \,.\, 2$ as follows, $\quad a^3bcdefghi,\ ab,\ ac,\ ad,\ ae,\ af,\ ag,\ ah,\ ai.$

(Observe that the partitionment is to be symmetrical as regards the letters which have a common index.) But, to take another example,

$$37 = 0 \,.\, 11 + 3 \,.\, 8 + 1 \,.\, 5 + 4 \,.\, 2 = 1 \,.\, 11 + 0 \,.\, 8 + 4 \,.\, 5 + 3 \,.\, 2.$$

The first of these gives the product $(abc)^8 d^5 (efgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the second type; and the second gives the product $a^{11}(bcde)^5(fgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the first type; viz., the partitions of the two products respectively are:

first product on second type, $(abc)^2 defgh,\ abcde,\ abcdf,\ abcdg,\ abcdh,\ ab,\ ac,\ bc$;

second product on first type, $a^2bcdefg,\ a^2bcdefh,\ a^2bcdegh,\ abcde,\ ab,\ ac,\ ad,\ ae$;

so that in the first example the type is sibi-reciprocal, but in the second example there are two conjugate types. Other examples are:

Parts	48	54	55	56	53		55		No.
2	14	3	1	0	3	6	0	2	Reciprocals.
5	0	2	3	0	6	0	5	0	
8	0	3	2	7	0	1	2	5	
11	0	0	2	0	0	3	0	1	
14	0	1	0	0	0	0	1	0	
17	0	0	0	0	1	0	0	0	
20	1	0	0	0	0	0	0	0	

viz., the first four columns give each of them a sibi-reciprocal type, but the last two double columns give conjugate types. It is required to investigate the general solution.

3139. (By J. J. Walker, F.R.S.)—Find what two relations must hold among Dr. Salmon's invariants A, B, C of the sextic $(a \ldots g)(x, y)^6$ when it is a perfect square.

3146. (By Professor Crofton, F.R.S.)—Show that, if the value of the expression $e^{\frac{1}{2}hD^2}F(x)$ be known, that of $e^{\frac{1}{2}hD^2}e^{\frac{1}{2}kx^2}F(x)$ can be found. If the value of the former be written $\Phi(h, x)$, the value of the latter will be $\quad (1-hk)^{-\frac{1}{2}}\, e^{\frac{\frac{1}{2}kx^2}{1-hk}}\, \Phi\left(\dfrac{h}{1-hk},\ \dfrac{x}{1-hk}\right).$

3147. (By the late T. Cotterill, M.A.)—Prove the two following theorems:—

1. Curves of the order n pass through the $\frac{1}{2}n(n-1)$ points of intersection of n lines, and the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ other lines, if the $(2n+1)$ lines are tangents to the same conic.

2. One curve of the order n passes through the $\frac{1}{2}n(n+1)$ points of inter-

section of $(n+1)$ lines, and the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ other lines, if the $(2n+2)$ lines are tangents to the same conic.

3202. (By Artemas Martin, LL.D.)—If three dice be piled up at random on a horizontal plane, what is the probability that the pile will not fall down?

3216. (By Artemas Martin, LL.D.)—A sphere is cut by a random plane, and then cut again: find the chance that the last section is a complete circle.

3223. (By W. Siverly.)—Find the greatest cube which can be cut from the solid generated by the revolution of a cycloid around its base.

3225. (By Artemas Martin, LL.D.)—Find four positive whole numbers, the sum of any two of which shall be a rational cube.
[2080913082956455142636, 4937801347510680732948,
7262810476410016163052, 2149721086932415893409848
are *one* set of numbers satisfying the conditions of the problem.]

3226. (By I. H. Turrell.)—In a quadrilateral which has no re-entrant angle, it is required to draw four circles, each touching two adjacent sides of the quadrilateral, and each circle and its opposite touching the two adjacent circles.

3227. (By J. B. Sanders.)—Given the velocity, distance, and direction of projection, when the force is attractive and varies as the distance: to find the orbit.

3228. (By Professor Wolstenholme, Sc.D.)—

1. In a tetrahedron ABCD, if AB be perpendicular to CD, and AC to BD, so also will AD be perpendicular to BC. Such a tetrahedron may be called rectangular, and we shall have $BC^2+AD^2=CA^2+BD^2=AB^2+CB^2$.

2. In a rectangular tetrahedron, the perpendiculars from the angular points on the opposite faces meet in a point, the centre of perpendiculars. Such a point does not exist for any other tetrahedron.

3. The three straight lines, each intersecting at right angles a pair of opposite edges, meet in the centre of perpendiculars. These three straight lines will also meet in a point in any equifacial tetrahedron.

4. There exists a *polar sphere* to any rectangular tetrahedron, that is, each angular point is the pole of the opposite face with respect to a sphere: the centre of this sphere is the centre of perpendiculars. The sphere will not be possible unless all the plane angles, containing one of the solid angles, are obtuse.

5. One sphere can be drawn through the middle points of the six edges, and through the feet of the shortest distances between opposite edges, and the centre of this sphere is the centre of inertia of the tetrahedron, which bisects the distance between the centre of the circumscribed sphere and the centre of perpendiculars. (First twelve points' sphere.)

6. The sphere which passes through the centres of inertia of the faces will also pass through their centres of perpendiculars; and these points in any face will be ends of a diameter of the circle in which that face cuts the sphere. This sphere also trisects the part of any perpendicular of the tetrahedron intercepted between the centre of perpendiculars and the corresponding angular point. (Second twelve points' sphere.)

7. These spheres and the circumscribed sphere have a common radical plane, and to this system of spheres belongs also the sphere of which the

centre of inertia and the centre of perpendiculars are ends of a diameter.

8. If R, ρ be the radii of the circumscribed sphere and polar sphere, and δ the distance between their centres, $\delta^2 = R^2 + 3\rho^2$; and the radius of the first twelve points' sphere is $\frac{1}{2}(R^2-\rho^2)^{\frac{1}{2}}$, that of the second is $\frac{1}{3}R$, and that of the sphere in (7) is $\frac{1}{4}(R^2+3\rho^2)^{\frac{1}{2}}$. The radius of the common section of all these spheres is $\frac{\rho}{\delta}(R^2-\rho^2)^{\frac{1}{2}}$.

9. If R, ρ' be the radii of the circumscribed and first twelve points, sphere, and δ' the distance between their centres, $\delta'^2 = R^2-\rho'^2$.

10. The centres of similarity of the circumscribed and second twelve points' sphere are the centres of inertia and of perpendiculars of the tetrahedron; whence, if $ALA'a$ be drawn through the centre of perpendiculars to meet the opposite face in A' and the circumscribed sphere in a, $La = 3\,LA'$.

3252. (By the late T. Cotterill, M.A.)—Prove the following theorems, in the enunciation of which a curve (simple or compound) of the order a is denoted by C_a :—

1. If of the $(a+b)\times p$ points of intersection of two curves C_{a+b} and C_p, $a\times p$ are on a curve C_a, the remaining $b\times p$ points are on a curve C_b.

2. If two curves C_a and C_p pass through a points on a curve C_k, then a curve C_{a+b} through the remaining $(a\times p-a)$ intersections will cut the curve C_p in $(bp+a)$ points lying on a curve C_{b+k}, which will cut the curve C_p again in $(k\times p-a)$ points on a curve C_k, which passes, or can be made to pass, through the a points from which we started.

3276. (By Artemas Martin, LL.D.)—Find how many different square numbers can be made with the 9 digits, using all the digits once (and only once) in each number? and what are they.

[157326849, = $(12543)^2$, is *one* such number.]

3302. (By W. Siverly.)—Required the axes of the three greatest equal ellipses that can be drawn in an ellipse whose axes are 2A and 2B.

3304. (By Professor Cayley, F.R.S.)—The coordinates x, y, z being proportional to the perpendicular distances from the sides of an equilateral triangle, it is required to trace the curve

$$(y-z)\sqrt{x}+(z-x)\sqrt{y}+(x-y)\sqrt{z}=0.$$

[Prof. Cayley remarks that the curve in question is a particular case of that which presents itself in the following theorem, communicated to him (with a demonstration) several years ago by Mr. J. Griffiths :—

The locus of a point (x, y, z) such that its pedal circle (that is, the circle which passes through the feet of the perpendiculars drawn from the point in question upon the sides of the triangle of reference) touches the nine-point circle, is the sextic curve

$$\left\{x\cos A\,(y\cos B-z\cos C)\left(\frac{y}{\cos B}-\frac{z}{\cos C}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{y\cos B\,(z\cos C-x\cos A)\left(\frac{z}{\cos C}-\frac{x}{\cos A}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{z\cos C\,(x\cos A-y\cos B)\left(\frac{x}{\cos A}-\frac{y}{\cos B}\right)\right\}^{\frac{1}{2}}=0.$$

It would be an interesting problem to trace this more general curve.]

3313. (By the late T. Cotterill, M.A.)—1. If a curve of order n pass

through $(n+1)$ points and the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ lines, then a curve of class n touches the $(n+1)$ lines and the $\frac{1}{2}n(n+1)$ lines connecting the $(n+1)$ points. Enunciate the analogous theorem in space.

2. Curves of order n can pass through the $\frac{1}{2}(n-1)n$ points of intersection of n lines and the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ other lines, provided the $(2n+1)$ lines are tangents to a conic. One curve of order n passes through the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ lines and the $\frac{1}{2}n(n+1)$ points of intersection of $(n+1)$ other lines, provided the $(2n+2)$ lines are tangents to a conic.

3326. (By J. B. Sanders.)—Find the velocity and periodic time of a body revolving in a circle at a distance of n radii from the earth's centre.

3327. (By W. Siverly.)—How many spheres, each 2 inches in diameter, will it take to fill a hollow globe 18 inches in diameter?

3395. (By Prof. Hudson, M.A.)—An indefinitely small piece of ice, the shape of which may be taken to be that of a right circular cylinder, is floating with its axis vertical in water. The part immersed receives deposits of ice in such a manner as to continue cylindrical, the radius and axis receiving equal increments in equal times. Find the ultimate shape of the part not immersed. If the specific gravity of ice be ·96, prove that the surface is formed by the revolution of $y^2(9x-y)^{25}=a^{27}$.

3412. (By Prof. Hudson, M.A.)—A pyramid on a square base, the altitude of the vertex being equal to a side of the base, consists of a very large number of steps, the treads being horizontal and the rises vertical. It is required to paint it. Find the area to be painted; and the cost, assuming that the cost of painting per square foot varies as the height; having given a side of the base and the cost of painting a square foot at the vertex.

3415. (By the Editor.)—A quadrilateral is inscribed in a circle, three of whose sides pass through fixed points; find the maximum and minimum values of the area of the quadrilateral.

3420. (By J. J. Walker, F.R.S.)—Find by what linear substitutions $(a, b, c \between xy)^2$ and $(a', b', c' \between xy)^2$ may be transformed simultaneously into $(\mathrm{A}, \mathrm{B}, \mathrm{C} \between xy)^2$ and [to a factor] $(\mathrm{A}, -\mathrm{B}, \mathrm{C} \between xy)^2$ respectively.

3437. (By the Rev. U. J. Knisely.)—A ground space, 56 feet square, is enclosed by a post fence 11 feet high; find the greatest number of saw-logs, each 2 feet in diameter and 12 feet long, which can be piled up in that space, not extending above the posts.

3438. (By G. O'Hanlon.)—Find when next the sun and planets will be in a straight line, and the interval between the occurring of the conjunction.

3481. (By Professor Cayley, F.R.S.)—Find, in the Hamiltonian form

$$\frac{d\eta}{dt}=\frac{d\mathrm{H}}{d\varpi},\quad \frac{d\varpi}{dt}=-\frac{d\mathrm{H}}{d\eta},\ \&c.,$$

the equations for the motion of a particle acted on by a central force.

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